Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.2c. Choice of the best model

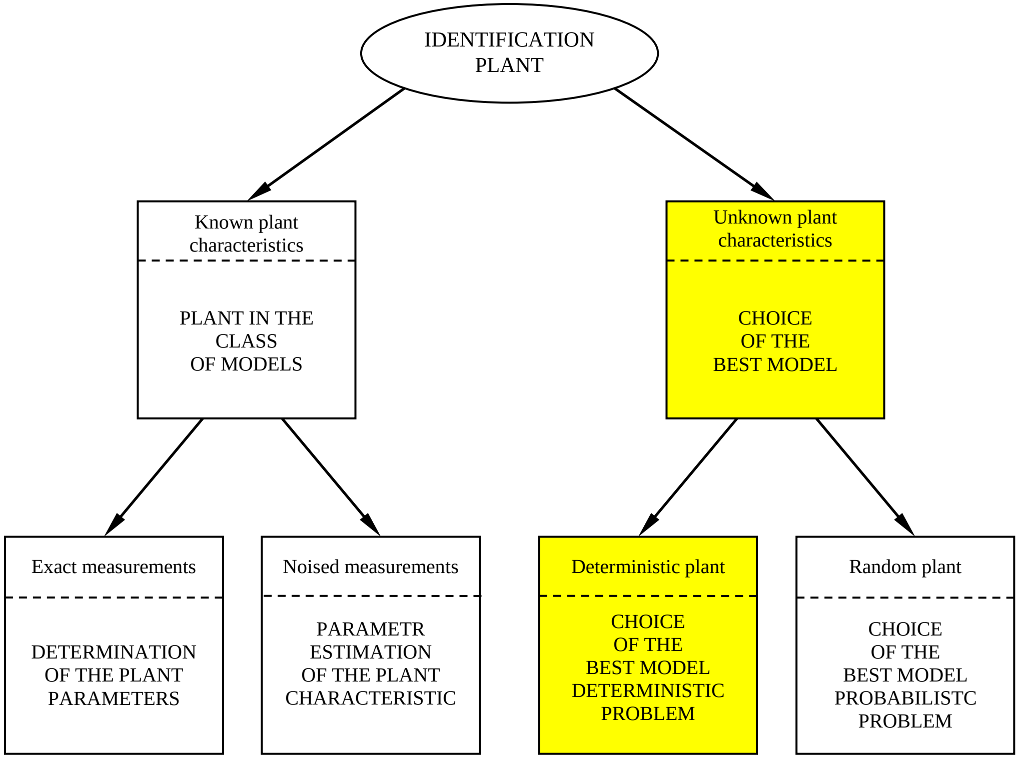
HUMAN CAPITAL
HUMAN – BEST INVESTMENT!

Wrocław University of Technology

EUROPEAN SOCIAL FUND

Project co-financed from the EU European Social Fund

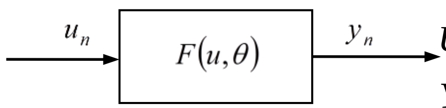
Typical identification tasks



HUMAN CAPITAL
HUMAN – BEST INVESTMENT!
Wrocław University of Technology
EUROPEAN SOCIAL FUND
Project co-financed from the EU European Social Fund

Plant in the class of model

D e t e r m i n i s t y c z n y

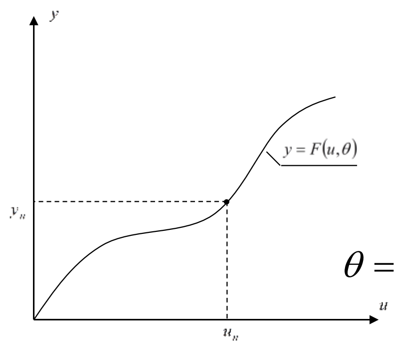


$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

$$y_n = F(u_n, \theta),$$
$$n = 1, 2, \ldots, N$$

$$Y_N = \overline{F}(U_N, \theta)$$

$$\theta = \overline{F}^{-1}(U_N, Y_N) \overset{df}{=} \Psi_N(U_N, Y_N)$$



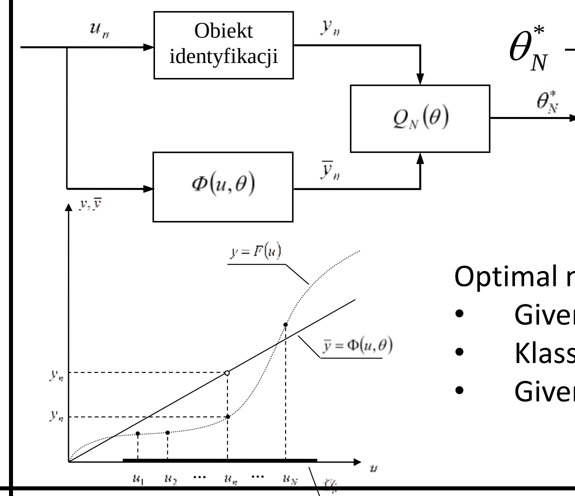
Choice of the best model

$$\bar{y} = \Phi(u, \theta_N^*)$$

$$\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$$

$$\theta_N^* = \Psi_N(U_N, Y_N)$$

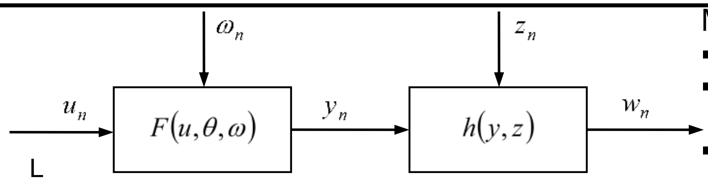
$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$



Optimal model for:

- Given measurements sequences
- Klass of models
- Given performance index

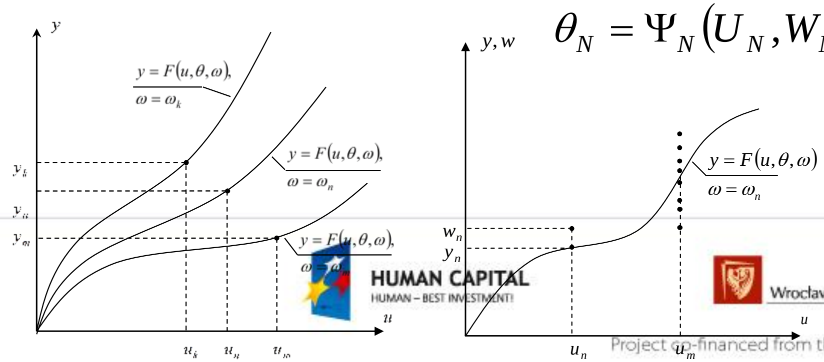
L o s o w y

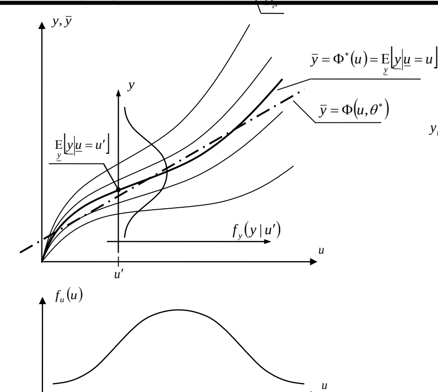


Methods:

- Least square
- Maximum likelihood
- Bayes'a

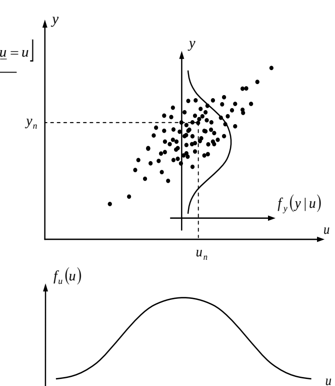
$$\theta_N = \Psi_N(U_N, W_N)$$





Full probabilistic knowledge

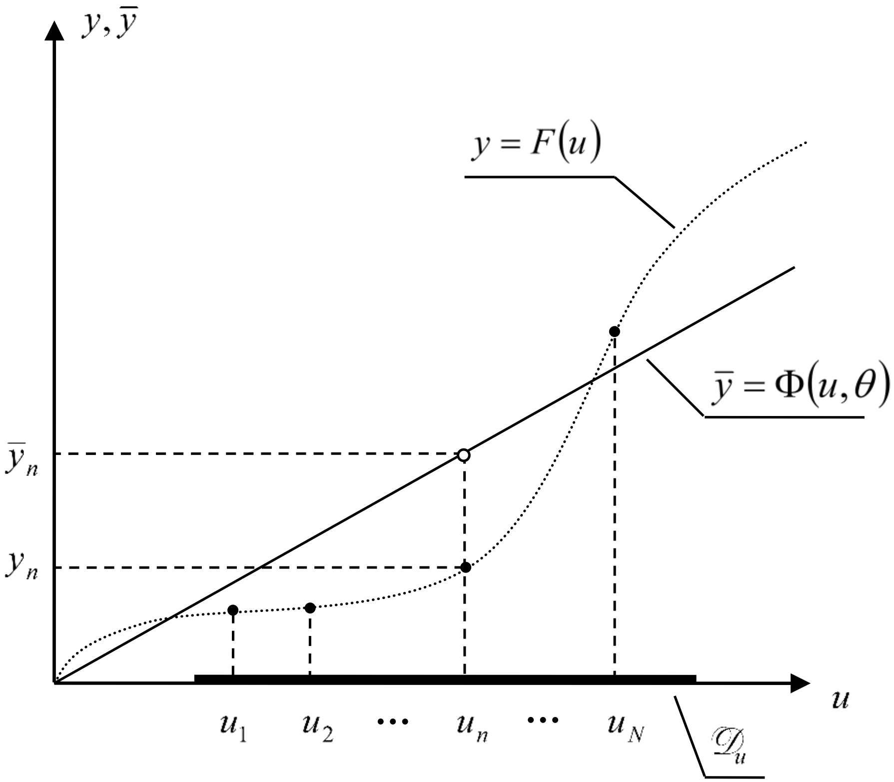
- First type regrsesion
- Second type regression

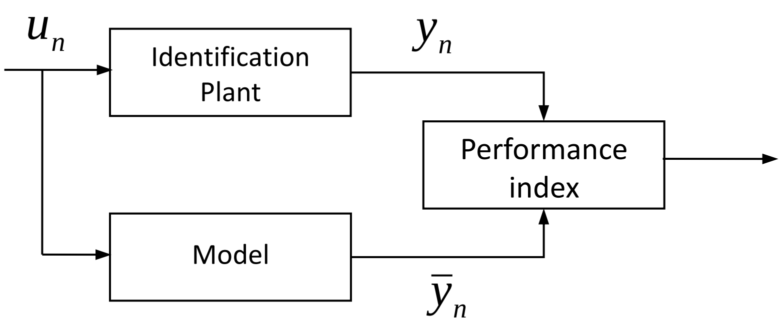


Unknown probabilostic knowledge

- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

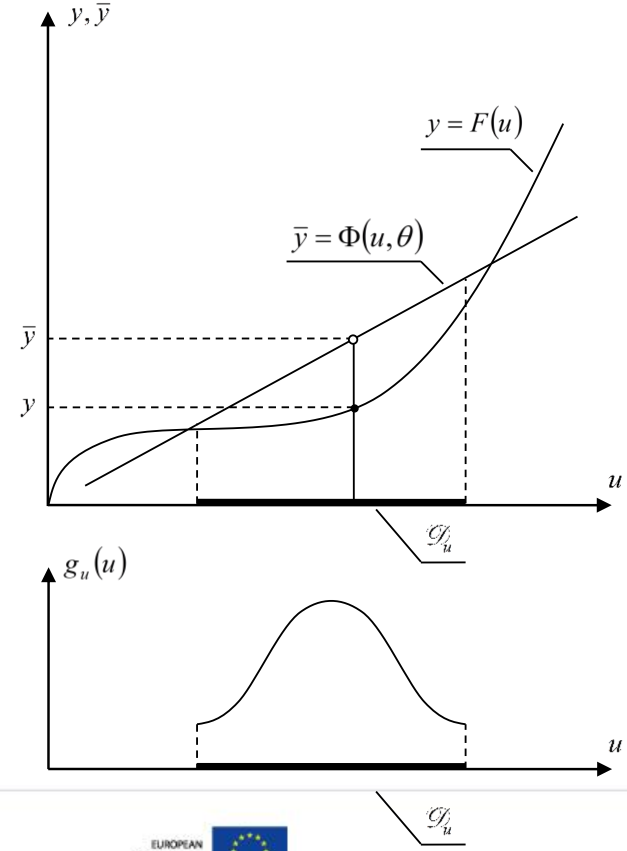
Choice of the best model
Deterministic problem





Approximation of static plant characteristic

- Problem formulation



Weight function: $g_u(u)$

Approximation of static plant characteristic

- Problem formulation

Static plant characteristic: $y = F(u)$

F – known function

u – input vector $\quad u \in \mathscr{U} \subseteq \mathscr{R}^S \quad \mathscr{U}$ – input domain

y – output vector $\quad y \in \mathscr{Y} \subseteq \mathscr{R}^L \quad \mathscr{Y}$ – output domain

Approximation function (model): $\bar{y} = \Phi(u, \theta)$

Φ – arbitrary given function

$\bar{y}$ – model output vector $\quad \bar{y} \in \overline{\mathscr{Y}} \subseteq \mathscr{R}^L \quad \mathscr{D}_u$ – subset of input domain

θ – vector of model parameters $\quad \theta \in \Theta \subseteq \mathscr{R}^R \quad \mathscr{Y}$ – output domain

HUMAN CAPITAL
HUMAN – BEST INVESTMENT!
Wrocław University of Technology
EUROPEAN SOCIAL FUND
Project co-financed from the EU European Social Fund

Approximation of static plant characteristic

- Problem formulation

Measure of difference: $\forall u \in \mathcal{D}_u \quad q(y,\bar{y}) = q(F(u),\Phi(u,\theta))$

where e. g.: $q(y,\bar{y}) = \sum_{l=1}^{L} \left(y^{(l)} - \bar{y}^{(l)}\right)^2 \quad q(y,\bar{y}) = \sum_{l=1}^{L} \left|y^{(l)} - \bar{y}^{(l)}\right|$ L – dimensional output

$$q(y,\bar{y}) = (y-\bar{y})^2 = (F(u)-\Phi(u,\theta))^2$$

$$q(y,\bar{y}) = |y-\bar{y}| = |F(u)-\Phi(u,\theta)|$$

one dimensional output $(L=1)$

Approximation of static plant characteristic

- Problem formulation

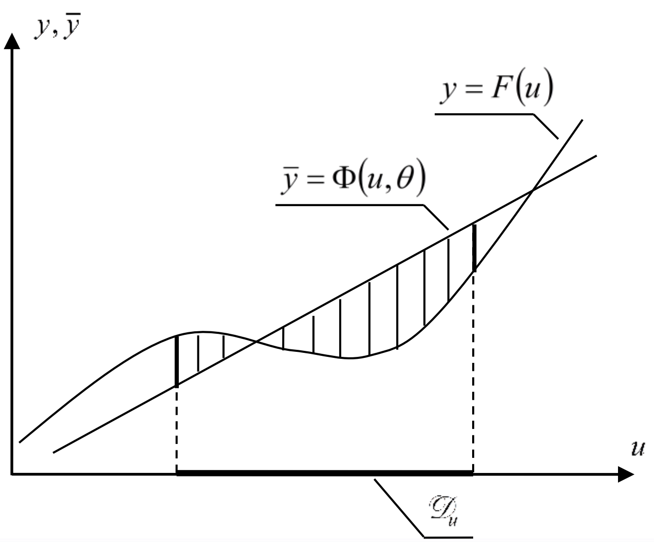
Performance index – a measure of the difference between function $F(u)$ and $\Phi(u,\theta)$

$$Q(\theta)=\left\|F(u)-\Phi(u,\theta)\right\|_{\mathcal{D}_u}$$

$$Q(\theta)=\int_{\mathcal{D}_u} q\big(F(u),\Phi(u,\theta)\big)\, g_u(u)\,du$$

e. g. :

$$Q(\theta)=\int_{\mathcal{D}_u} \left|F(u)-\Phi(u,\theta)\right|\, g_u(u)\,du$$



Approximation of static plant characteristic

- Problem formulation

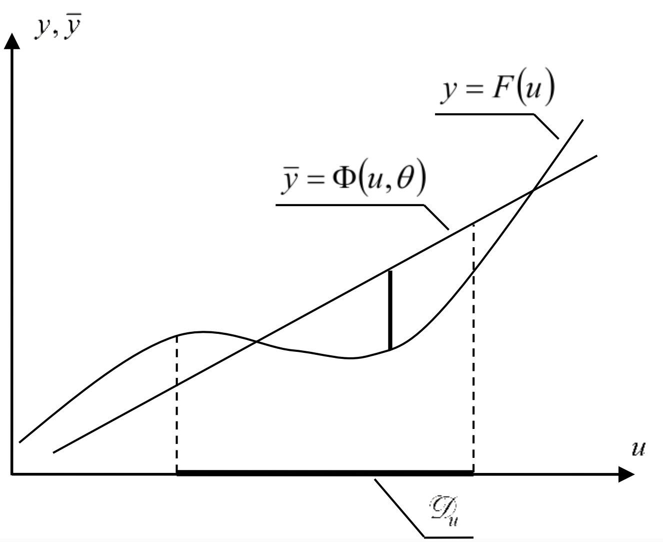
Performance index – a measure of the difference between function $F(u)$ and $\Phi(u,\theta)$

$$Q(\theta) = \left\| F(u) - \Phi(u,\theta) \right\|_{\mathcal{D}_u}$$

$$Q(\theta) = \max_{u \in \mathcal{D}_u} \left\{ q\left(F(u), \Phi(u,\theta)\right) g_u(u) \right\}$$

e. g. :

$$Q(\theta) = \max_{u \in \mathcal{D}_u} \left\{ \left| F(u) - \Phi(u,\theta) \right| g_u(u) \right\}$$

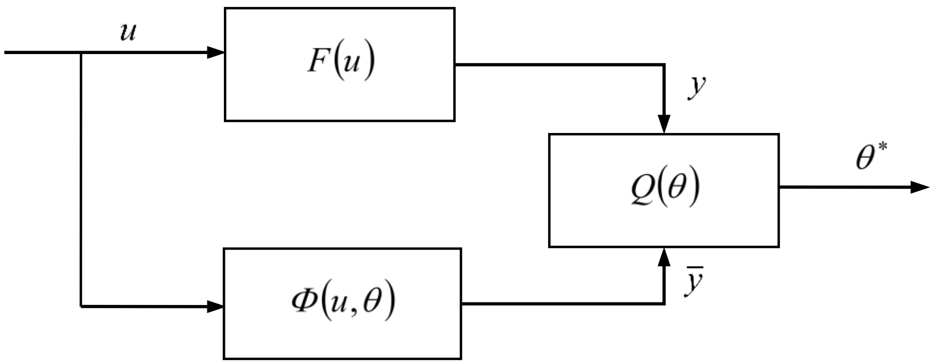


Approximation of static plant characteristic

- Problem formulation

Optimal model: $\bar{y} = \Phi(u, \theta^*)$

where θ^* – optimal model parameters: $\theta^* \rightarrow \quad Q(\theta^*) = \min_{\theta \in \Theta} Q(\theta)$



Linear case – problem formulation

Model: $\bar{y} = \Phi(u,\theta) = \theta^T \varphi(u)$

where $\theta \overset{df}{=} \begin{bmatrix} \theta^{(1)} \\ \theta^{(2)} \\ \vdots \\ \theta^{(R)} \end{bmatrix}$ $\varphi(u) \overset{df}{=} \begin{bmatrix} \varphi_1(u) \\ \varphi_2(u) \\ \vdots \\ \varphi_R(u) \end{bmatrix}$ e. g. : $\varphi(u) = \begin{bmatrix} 1 \\ u \\ \vdots \\ u^{R-1} \end{bmatrix}$

Performance index:

$$Q(\theta) = \int_{\mathcal{D}_u} \left(F(u) - \theta^T \varphi(u)\right)^2 g_u(u)\, du$$

Linear case - solution

Optimality condition for the vector θ :

$$\underset{\theta}{\operatorname{grad}} Q(\theta) = -2\left[\int_{\mathscr{D}_u} \left(F(u) - \theta^{*T}\varphi(u)\right)\varphi(u) g_u(u)\, du\right] = 0_R, \qquad 0_R = \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} R$$

The solution is:

$$\theta^* = \left[\int_{\mathscr{D}_u} \varphi(u)\,\varphi^T(u)\, g_u(u)\, du\right]^{-1} \int_{\mathscr{D}_u} F(u)\,\varphi(u) g_u(u)\, du$$

Linear case - solution

Note that:

$$\int_{\mathcal{D}_u} \varphi(u)\,\varphi^T(u)\,g_u(u)\,du =$$

$$\begin{bmatrix} \int_{\mathcal{D}_u} \varphi_1(u)\,\varphi_1(u) g_u(u)\,du & \int_{\mathcal{D}_u} \varphi_1(u)\,\varphi_2(u) g_u(u)\,du & \cdots & \int_{\mathcal{D}_u} \varphi_1(u)\,\varphi_R(u) g_u(u)\,du \\ \int_{\mathcal{D}_u} \varphi_2(u)\,\varphi_1(u) g_u(u)\,du & \int_{\mathcal{D}_u} \varphi_2(u)\,\varphi_2(u) g_u(u)\,du & \cdots & \int_{\mathcal{D}_u} \varphi_2(u)\,\varphi_R(u) g_u(u)\,du \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathcal{D}_u} \varphi_R(u)\,\varphi_1(u) g_u(u)\,du & \int_{\mathcal{D}_u} \varphi_R(u)\,\varphi_2(u) g_u(u)\,du & \cdots & \int_{\mathcal{D}_u} \varphi_R(u)\,\varphi_R(u) g_u(u)\,du \end{bmatrix}$$

Linear case - solution

Let us choose $\varphi_i(u)$ such that:

$$\int_{\mathcal{D}_u} \varphi_i(u)\,\varphi_j(u)\,g_u(u)\,du = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \quad , \quad i, j = 1, 2, \ldots, R$$

Then: $$\int_{\mathcal{D}_u} \varphi(u)\,\varphi^T(u)\,g_u(u)\,du = \mathbf{I}$$

and approximation algorithm has the form:

$$\theta^* = \int_{\mathcal{D}_u} F(u)\,\varphi(u) g_u(u)\,du$$

Linear case - example

Plant characteristic: $y = F(u) = u^2$

Model: $\bar{y} = \Phi(u,\theta) = \theta^T \varphi(u) = \theta^{(1)} + \theta^{(2)} u$ where: $\varphi(u) = \begin{bmatrix} 1 \\ u \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta^{(1)} \\ \theta^{(2)} \end{bmatrix}$

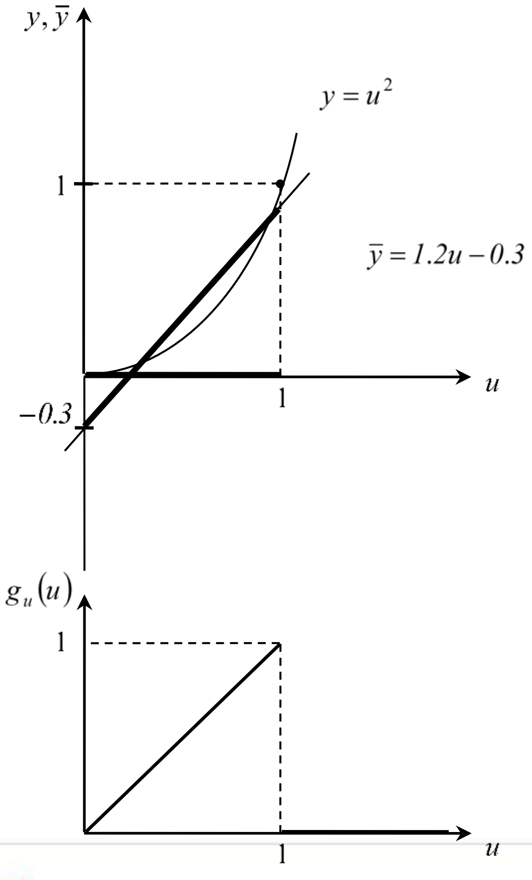
Interval: $\mathscr{D}_u = \{u \in \mathscr{R} : 0 \le u \le 1\}$

Weight function: $g_u(u) = \begin{cases} u & \text{for} \quad u \in [0,1] \\ 0 & \text{for} \quad u \notin [0,1] \end{cases}$

Performance index: $Q(\theta) = \int\limits_{\mathscr{D}_u} \left(F(u) - \theta^T \varphi(u)\right)^2 g_u(u)\, du$

Linear case - example

Graphical interpretation:



Linear case - example

Performance index has the form:

$$Q(\theta)=\int_{\mathcal{D}_u}\left(F(u)-\theta^T\varphi(u)\right)^2 g_u(u)\,du=\int_0^1\left(u^2-\left[\theta^{(1)}\ \theta^{(2)}\right]\begin{bmatrix}1\\u\end{bmatrix}\right)^2 u\ du$$

Optimality condition has the form:

$$\operatorname*{grad}_{\theta} Q(\theta)=-2\left[\int_0^1\left(u^2-\left[\theta^{(1)*}\ \theta^{(2)*}\right]\begin{bmatrix}1\\u\end{bmatrix}\right)\begin{bmatrix}1\\u\end{bmatrix}u\ du\right]=0_2$$

Solution has the form:

$$\theta^*=\left[\int_0^1\begin{bmatrix}1\\u\end{bmatrix}\left[1\ u\right]u\ du\right]^{-1}\int_0^1 u^2\begin{bmatrix}1\\u\end{bmatrix}u\ du=\begin{bmatrix}\int_0^1 u\ du & \int_0^1 u^2\ du\\ \int_0^1 u^2\ du & \int_0^1 u^3\ du\end{bmatrix}^{-1}\begin{bmatrix}\int_0^1 u^3\ du\\ \int_0^1 u^4\ du\end{bmatrix}$$

Linear case - example

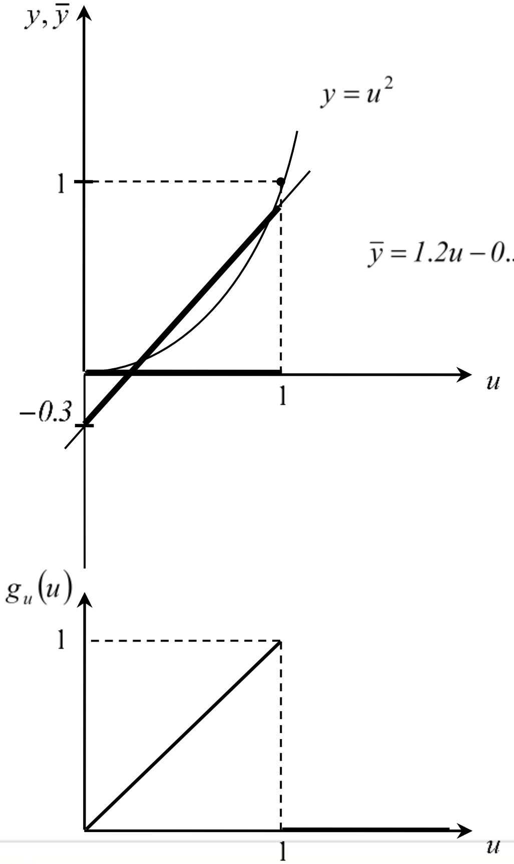
Consequently:

$$\theta^* = \begin{bmatrix} \theta^{(1)*} \\ \theta^{(2)*} \end{bmatrix} = \begin{bmatrix} \dfrac{\int_0^1 u^3\,du \times \int_0^1 u^3\,du - \int_0^1 u^4\,du \times \int_0^1 u^2\,du}{\int_0^1 u\,du \times \int_0^1 u^3\,du - \left(\int_0^1 u^2 du\right)^2} \\ \dfrac{\int_0^1 u^4\,du \times \int_0^1 u\,du - \int_0^1 u^3\,du \times \int_0^1 u^2\,du}{\int_0^1 u\,du \times \int_0^1 u^3\,du - \left(\int_0^1 u^2 du\right)} \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{1}{4}u^4\Big|_0^1 \times \frac{1}{4}u^4\Big|_0^1 - \frac{1}{5}u^5\Big|_0^1 \times \frac{1}{3}u^3\Big|_0^1}{\frac{1}{2}u^2\Big|_0^1 \times \frac{1}{4}u^4\Big|_0^1 - \left(\frac{1}{3}u^3\Big|_0^1\right)^2} \\ \dfrac{\frac{1}{5}u^5\Big|_0^1 \times \frac{1}{2}u^2\Big|_0^1 - \frac{1}{4}u^4\Big|_0^1 \times \frac{1}{3}u^3\Big|_0^1}{\frac{1}{2}u^2\Big|_0^1 \times \frac{1}{4}u^4\Big|_0^1 - \left(\frac{1}{3}u^3\Big|_0^1\right)^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{1}{4} \times \frac{1}{4} - \frac{1}{5} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{4} - \left(\frac{1}{3}\right)^2} \\ \dfrac{\frac{1}{5} \times \frac{1}{2} - \frac{1}{4} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{4} - \left(\frac{1}{3}\right)^2} \end{bmatrix} = \begin{bmatrix} -0.3 \\ 1.2 \end{bmatrix}$$

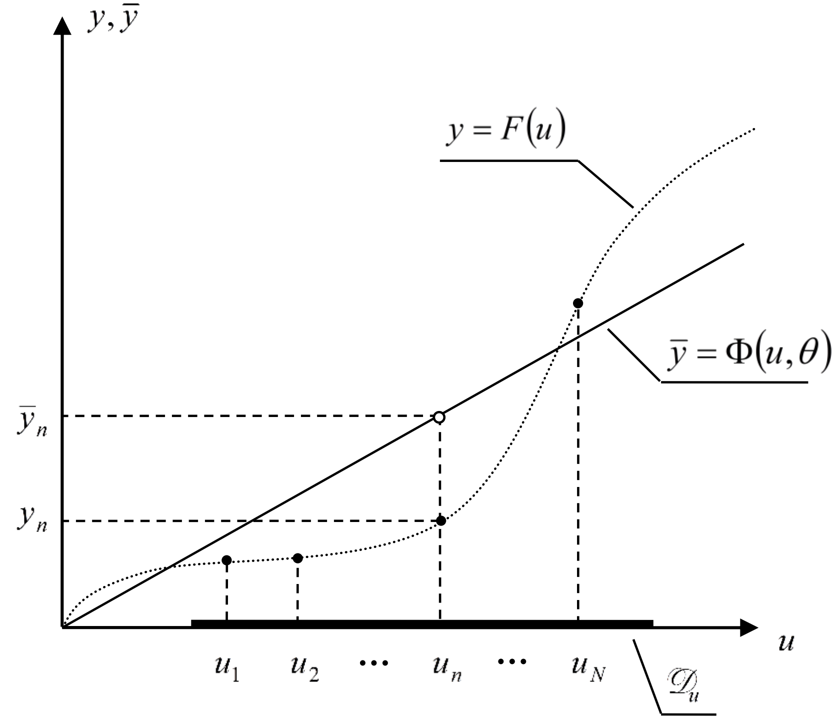
Linear case - example

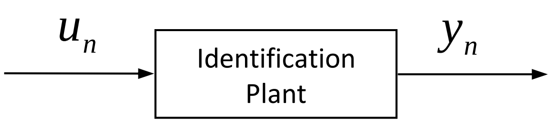
Optimal model: $\bar{y} = 1.2\,u - 0.3$

Graphical interpretation:



Choice of the best model based on the noise free measurements





Choice of the best model based on the noise free measurements

- Problem formulation

Experiment: $U_N = [u_1 \quad u_2 \quad \cdots \quad u_N], \quad Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$

Approximation function (model): $\bar{y} = \Phi(u, \theta)$

Measure of the difference: $\forall n = 1, 2, \ldots, N \quad q(y_n, \bar{y}_n) = q(y_n, \Phi(u_n, \theta))$

Choice of the best model based on the noise free measurements

- Problem formulation

Performance index: $Q_N(\theta) = \left\| Y_N - \bar{Y}_N(\theta) \right\|_{U_N}$

where: $\bar{Y}_N(\theta) \overset{df}{=} \left[\Phi(u_1,\theta) \quad \Phi(u_2,\theta) \quad \cdots \quad \Phi(u_N,\theta) \right]$

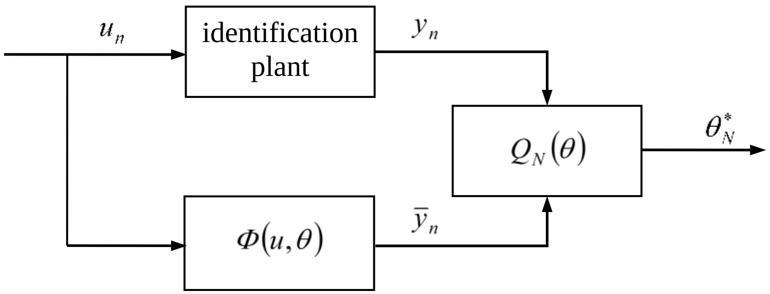
$$Q_N(\theta) = \sum_{n=1}^{N} \alpha_n q(y_n, \bar{y}_n) = \sum_{n=1}^{N} \alpha_n q(y_n, \Phi(u_n,\theta)) \quad \text{e. g. :} \quad Q_N(\theta) = \sum_{n=1}^{N} |y_n - \bar{y}_n| = \sum_{n=1}^{N} |y_n - \Phi(u_n,\theta)|$$

$$Q_N(\theta) = \max_{1\le n\le N} \{ q(y_n, \bar{y}_n) \} = \max_{1\le n\le N} \{ q(y_n, \Phi(u_n,\theta)) \} \quad \text{e. g. :} \quad Q_N(\theta) = \max_{1\le n\le N} \{ |y_n - \bar{y}_n| \} = \max_{1\le n\le N} \{ |y_n - \Phi(u_n,\theta)| \}$$

Choice of the best model based on the noise free measurements

- Problem formulation

Optimal model: $\bar{y} = \Phi(u, \theta_N^*)$ $\theta_N^* \rightarrow Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$



The model is optimal for:

- given measurement sequence
- proposed model
- performance index

Approximation task

$$y = F(u) \qquad \bar{y} = \Phi(u, \theta^*)$$

where: θ^* – optimal vector parameter:

$$\theta^* \rightarrow \quad Q(\theta^*) = \min_{\theta \in \Theta} Q(\theta)$$

$$Q(\theta) = \int_{\mathcal{D}_u} q(F(u), \Phi(u, \theta))\, g_u(u)\, du$$

Identification task

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N], \quad Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N] \qquad \bar{y} = \Phi(u, \theta_N^*)$$

where: θ_N^* – optimal vector parameter:

$$\theta_N^* \rightarrow Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$$

$$Q_N(\theta) = \sum_{n=1}^{N} \alpha_n q(y_n, \bar{y}_n) = \sum_{n=1}^{N} \alpha_n q(y_n, \Phi(u_n, \theta))$$

$$\theta_N^* \sim \theta^* \qquad \theta_N^* \xrightarrow[U_N, \alpha_n \sim g_u(u)]{} \theta^*$$

HUMAN CAPITAL
HUMAN – BEST INVESTMENT!

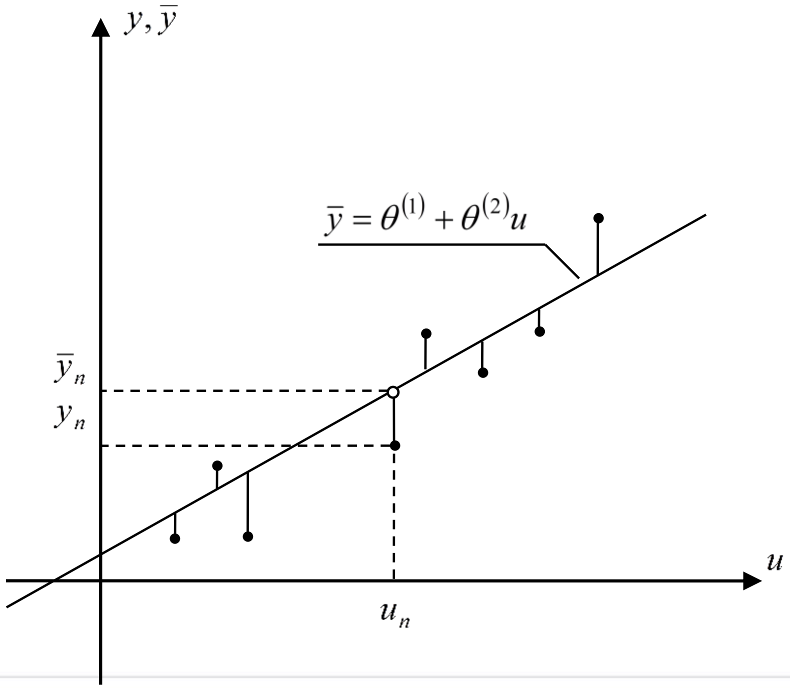
Wrocław University of Technology

EUROPEAN SOCIAL FUND

Project co-financed from the EU European Social Fund

Linear case – example

Experiment: $U_N = [u_1 \quad u_2 \quad \cdots \quad u_N], \quad Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$



Model: $\bar{y} = \Phi(u, \theta) = \theta^T \varphi(u) = \theta^{(1)} + \theta^{(2)} u$

where $\varphi(u) = \begin{bmatrix} 1 \\ u \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta^{(1)} \\ \theta^{(2)} \end{bmatrix}$

Performance index:

$$Q_N(\theta) = \sum_{n=1}^{N} (y_n - \bar{y}_n)^2 = \sum_{n=1}^{N} \left(y_n - \theta^T \varphi(u_n)\right)^2$$

Linear case – example

Optimality condition has the form: $\underset{\theta}{\operatorname{grad}} Q_N(\theta) = -2\sum_{n=1}^{N}\left(y_n - \theta_N^{*T}\varphi(u_n)\right)\varphi(u_n) = 0_2$

Solution has the form: $\theta_N^* = \left[\sum_{n=1}^{N}\varphi(u_n)\varphi^T(u_n)\right]^{-1} \times \sum_{n=1}^{N} y_n\varphi(u_n)$

$$\theta_N^* = \begin{bmatrix}\theta_N^{*(1)} \\ \theta_N^{*(2)}\end{bmatrix} = \begin{bmatrix} \dfrac{\left(\sum_{n=1}^{N}u_n^2\right)\times\left(\sum_{n=1}^{N}y_n\right) - \left(\sum_{n=1}^{N}u_n\right)\times\left(\sum_{n=1}^{N}y_n u_n\right)}{N\sum_{n=1}^{N}u_n^2 - \left(\sum_{n=1}^{N}u_n\right)\times\left(\sum_{n=1}^{N}y_n u_n\right)} \\ \dfrac{N\sum_{n=1}^{N}y_n u_n - \left(\sum_{n=1}^{N}u_n\right)\times\left(\sum_{n=1}^{N}y_n\right)}{N\sum_{n=1}^{N}u_n^2 - \left(\sum_{n=1}^{N}u_n\right)\times\left(\sum_{n=1}^{N}y_n u_n\right)} \end{bmatrix}$$

Linear case - recursive algorithm

$$Q_N(\theta)=\sum_{n=1}^{N}(y_n-\bar{y}_n)^2=\sum_{n=1}^{N}\left(y_n-\theta^T\varphi(u_n)\right)^2$$

For N measurements $U_N=[u_1\quad u_2\quad \cdots\quad u_N],\quad Y_N=[y_1\quad y_2\quad \cdots\quad y_N]$

$$\theta_N^*=\left[\sum_{n=1}^{N}\varphi(u_n)\varphi^T(u_n)\right]^{-1}\times\sum_{n=1}^{N}y_n\varphi(u_n)$$

New *N+1* measurement point $u_{N+1},\, y_{N+1}$

How to adopt vector of parameters using new measurement?

$$\theta_{N+1}^*=A\left(\theta_N^*,u_{N+1},y_{N+1}\right)$$

$$Q_{N+1}(\theta)=\sum_{n=1}^{N+1}(y_n-\bar{y}_n)^2=\sum_{n=1}^{N+1}\left(y_n-\theta^T\varphi(u_n)\right)^2$$

$$\theta_{N+1}^*=\left[\sum_{n=1}^{N+1}\varphi(u_n)\varphi^T(u_n)\right]^{-1}\times\sum_{n=1}^{N+1`}y_n\varphi(u_n)=$$

$$\left[\sum_{n=1}^{N}\varphi(u_n)\varphi^T(u_n)+\varphi(u_{N+1})\varphi^T(u_{N+1})\right]^{-1}\times\left[\sum_{n=1}^{N`}y_n\varphi(u_n)+y_{N+1}\varphi(u_{N+1})\right]$$

Linear case - recursive algorithm

$$\left(\mathbf{A}+\mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\mathbf{B}\left(\mathbf{D}+\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1}$$

$\mathbf{B}=\varphi$ – wektor kolumnowy

$$\mathbf{D}^{-1}=1$$

$$\mathbf{C}=\varphi^{T}$$

$$\left(\mathbf{A}+\varphi\varphi^{T}\right)^{-1}=\mathbf{A}^{-1}-\mathbf{A}^{-1}\varphi\left(1+\varphi^{T}\mathbf{A}^{-1}\varphi\right)^{-1}\varphi^{T}\mathbf{A}^{-1}$$

$$\mathbf{D}^{-1}=-1$$

$$\left(\mathbf{A}-\varphi\varphi^{T}\right)^{-1}=\mathbf{A}^{-1}+\mathbf{A}^{-1}\varphi\left(1-\varphi^{T}\mathbf{A}^{-1}\varphi\right)^{-1}\varphi^{T}\mathbf{A}^{-1}$$

Linear case - recursive algorithm

Let: $$\varphi_n = \varphi(u_n),\; n = 1, 2, \ldots, N+1$$

$$\left(\sum_{n=1}^{N+1} \varphi_n \varphi_n^T\right)^{-1} = \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T + \varphi_{N+1}\varphi_{N+1}^T\right)^{-1} =$$

$$= \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T\right)^{-1} - \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T\right)^{-1} \frac{\varphi_{N+1}\varphi_{N+1}^T}{1 + \varphi_{N+1}^T \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T\right)^{-1} \varphi_{N+1}} \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T\right)^{-1}$$

Let: $$P_N = \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T\right)^{-1}$$

$$P_{N+1} = P_N - P_N \frac{\varphi_{N+1}\varphi_{N+1}^T}{1 + \varphi_{N+1}^T P_N \varphi_{N+1}} P_N$$

Linear case - recursive algorithm

$$\theta^*_{N+1} = \left(P_N - P_N \frac{\varphi_{N+1}\varphi^T_{N+1}}{1+\varphi^T_{N+1}P_N\varphi_{N+1}} P_N \right)\left(\sum_{n=1}^{N} \varphi_n y_n + \varphi_{N+1} y_{N+1} \right) =$$

$$= P_N \sum_{n=1}^{N} \varphi_n y_n + P_N \varphi_{N+1} y_{N+1} - \frac{P_N \varphi_{N+1}\varphi^T_{N+1} P_N}{1+\varphi^T_{N+1}P_N\varphi_{N+1}} \sum_{n=1}^{N} \varphi_n y_n - \frac{P_N \varphi_{N+1}\varphi^T_{N+1} P_N \varphi_{N+1} y_{N+1}}{1+\varphi^T_{N+1}P_N\varphi_{N+1}} =$$

$$= P_N \sum_{n=1}^{N} \varphi_n y_n + \frac{P_N \varphi_{N+1} y_{N+1} + P_N \varphi_{N+1} y_{N+1} \varphi^T_{N+1} P_N \varphi_{N+1} - P_N \varphi_{N+1} y_{N+1} \varphi^T_{N+1} P_N \varphi_{N+1} - P_N \varphi_{N+1}\varphi^T_{N+1} P_N \sum_{n=1}^{N} \varphi_n y_n}{1+\varphi^T_{N+1}P_N\kappa_{N+1}} =$$

$$= P_N \sum_{n=1}^{N} \varphi_n y_n + \frac{P_N \varphi_{N+1} y_{N+1} - P_N \varphi_{N+1}\varphi^T_{N+1} P_N \sum_{n=1}^{N} \varphi_n y_n}{1+\varphi^T_{N+1}P_N\varphi_{N+1}} y =$$

$$= P_N \sum_{n=1}^{N} \varphi_n y_n + \frac{P_N \varphi_{N+1} y_{N+1}}{1+\varphi^T_{N+1}P_N\varphi_{N+1}} \left(y_{N+1} + \varphi^T_{N+1} P_N \sum_{n=1}^{N} \varphi_n y_n \right)$$

$$= \theta^*_N + K_{N+1}\left(y_{N+1} - \varphi^T_{N+1}\theta_N \right)$$

$$K_{N+1} = \frac{P_N \varphi_{N+1}}{1+\varphi^T_{N+1}P_N\varphi_{N+1}}$$

where:

Linear case - recursive algorithm

Finaly:

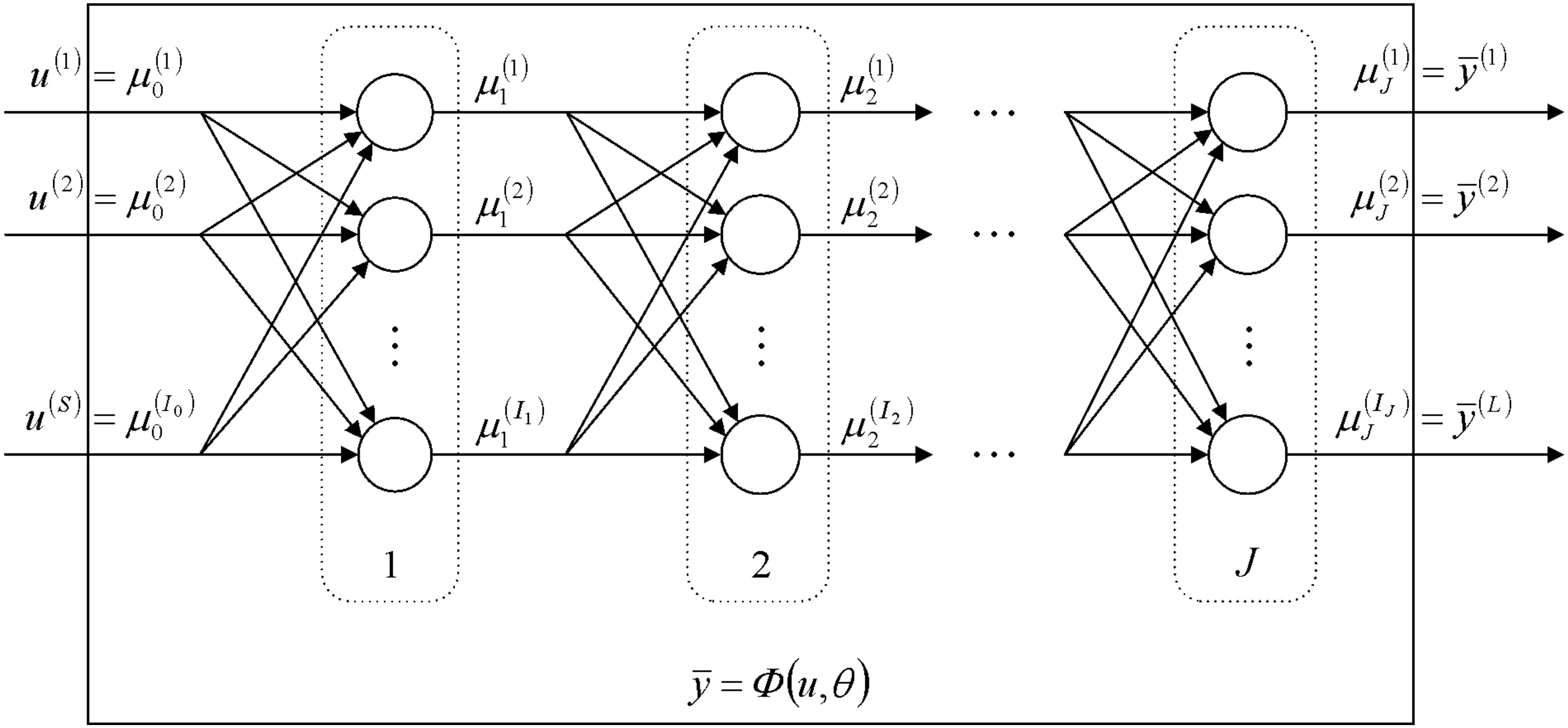
$$\theta_N^* = \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T \right)^{-1} \sum_{n=1}^{N} \varphi_n y_n = P_N \sum_{n=1}^{N} \varphi_n y_n \text{, gdzie: } P_N = \left(\sum_{n=1}^{N} \varphi_n \varphi_n^T \right)^{-1}$$

$$\theta_{N+1}^* = A\left(\theta_N^*, u_{N+1}, y_{N+1}\right) = \theta_N^* + K_{N+1}\left[y_{N+1} - \varphi(u_{N+1})^T \theta_N^* \right]$$

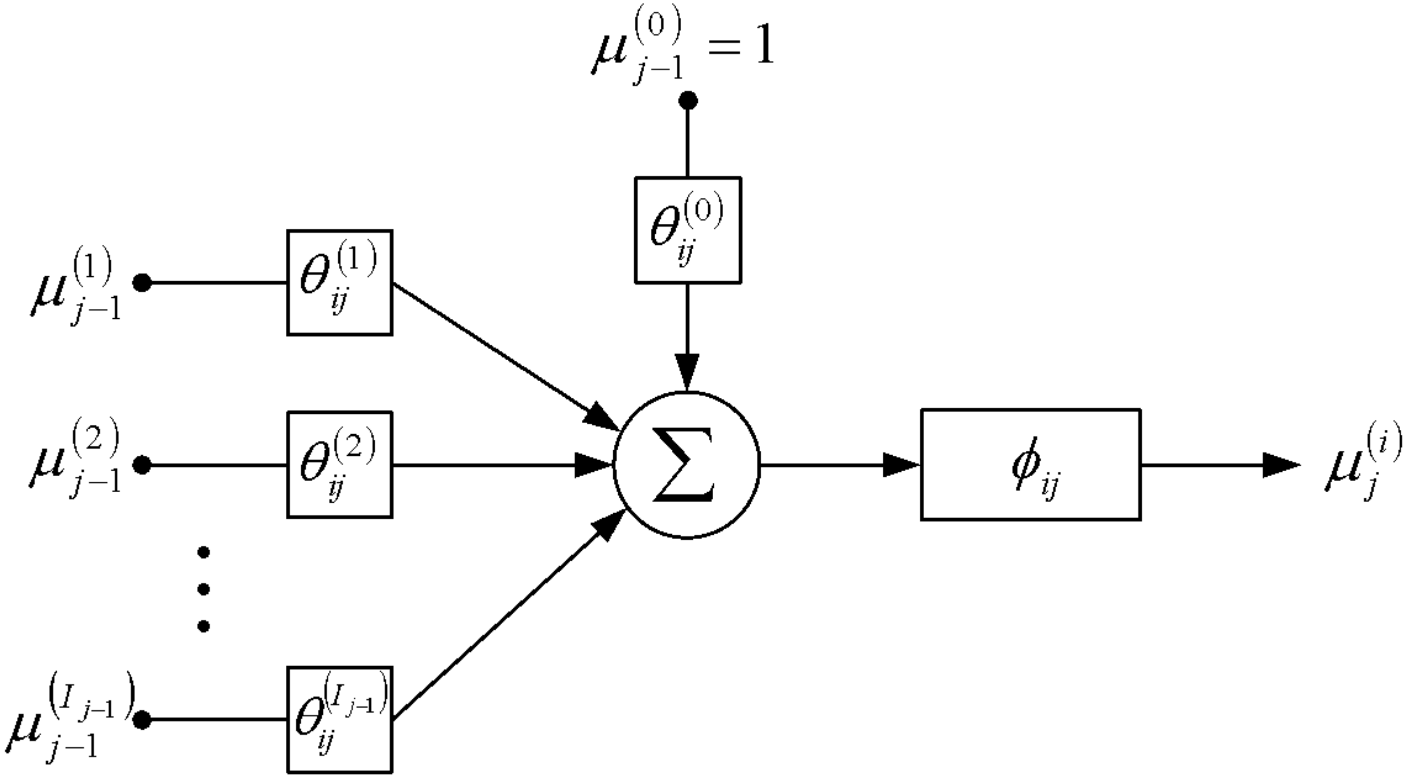
$$K_{N+1} = \frac{P_N \varphi(u_{N+1})}{1 + \varphi(u_{N+1})^T P_N \varphi(u_{N+1})}$$

$$P_{N+1} = P_N - \frac{P_N \varphi(u_{N+1}) \varphi(u_{N+1})^T P_N}{1 + \varphi(u_{N+1})^T P_N \varphi(u_{N+1})}$$

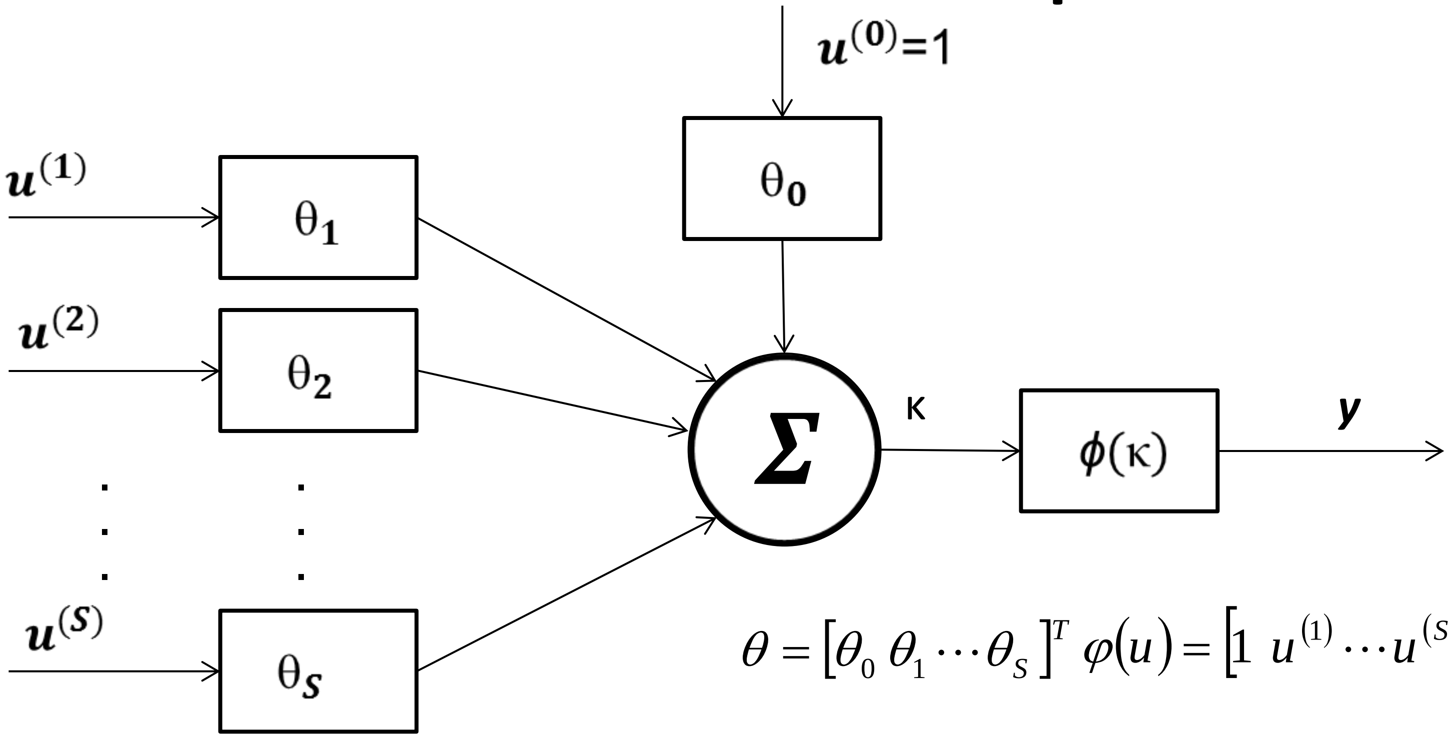
Neural networks



Neuron model



Neuron model simplification



$$\theta = [\theta_0 \ \theta_1 \cdots \theta_S]^T \quad \varphi(u) = [1 \ u^{(1)} \cdots u^{(S)}]^T$$

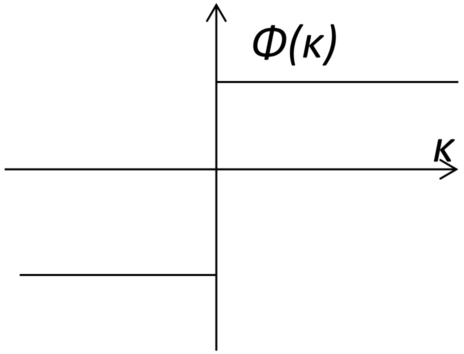
$$y = \phi\left(\sum_{s=1}^{S} \theta_s u^{(s)} + \theta_0\right) = \phi\left(\theta^T \varphi(u)\right)$$

Φ – activation function

Activation function

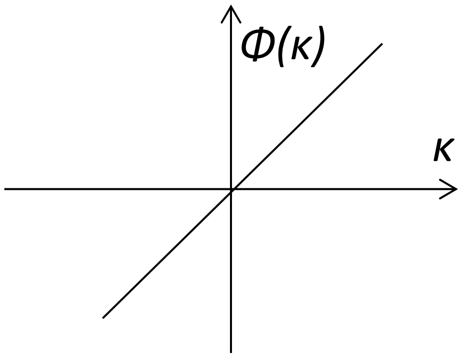
$$\phi(\kappa) = \begin{cases} 1 \; dla \;\; \kappa > 0 \\ -1 \; dla \;\; \kappa \le 0 \end{cases}$$

Perceptron



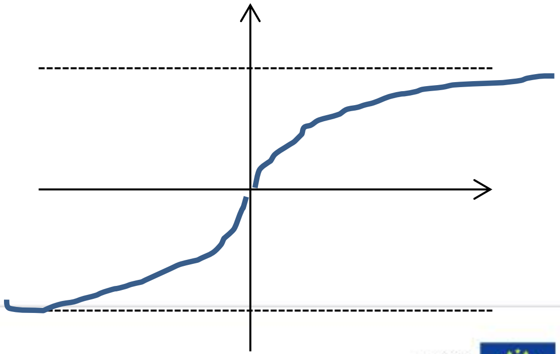
$$\phi(\kappa) = \kappa$$

Adaline
Adaptive Linear Neuron



$$\phi(\kappa) = \frac{1}{1 + e^{-\beta\kappa}}$$

$$\phi(\kappa) = \tanh(\beta\kappa) = \frac{1 - e^{-\beta\kappa}}{1 + e^{-\beta\kappa}}$$

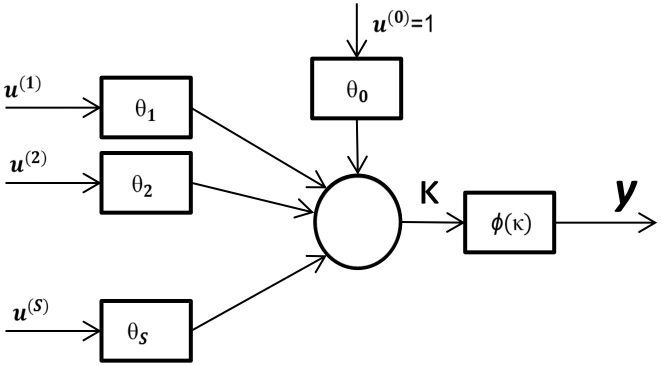


Sigmoidal Neuron

Neuron learning

Adaline
Adaptive Linear Neuron

$$\phi(\kappa) = \kappa$$



$$y = \phi\left(\sum_{s=1}^{S} \theta_s u^{(s)} + \theta_0\right) = \phi\left(\theta^T \varphi(u)\right) = \theta^T \varphi(u)$$

$$\theta = [\theta_0\ \theta_1 \cdots \theta_S]^T \qquad \varphi(u) = [1\ u^{(1)} \cdots u^{(S)}]^T$$

$$[u_1\ u_2 \cdots u_N] = U_N \qquad [y_1\ y_2 \cdots y_N] = Y_N$$

$$Q_N(\theta) = \sum_{n=1}^{N} (y_n - \bar{y}_n)^2 = \sum_{n=1}^{N} \left(y_n - \theta^T \varphi(u_n)\right)^2$$

$$\theta_N^* = \left[\sum_{n=1}^{N} \varphi(u_n) \varphi^T(u_n)\right]^{-1} \times \sum_{n=1}^{N} y_n \varphi(u_n)$$

Or recursive algorithm

Recursive algorithm

$$\theta^*_{N+1} = A\left(\theta^*_N, u_{N+1}, y_{N+1}\right) = \theta^*_N + K_{N+1}\left[y_{N+1} - \varphi\left(u_{N+1}\right)^T \theta^*_N\right]$$

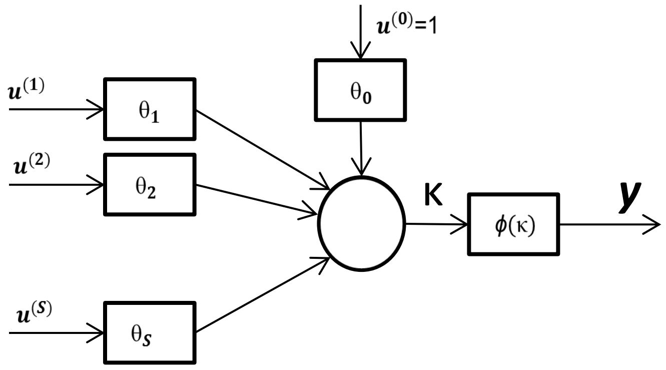
$$K_{N+1} = \frac{P_N \varphi\left(u_{N+1}\right)}{1 + \varphi\left(u_{N+1}\right)^T P_N \varphi\left(u_{N+1}\right)}$$

$$P_{N+1} = P_N - \frac{P_N \varphi\left(u_{N+1}\right) \varphi\left(u_{N+1}\right)^T P_N}{1 + \varphi\left(u_{N+1}\right)^T P_N \varphi\left(u_{N+1}\right)}$$

Neuron learning

Like neuron Adaline (Adaptive Linear Neuron)

$\phi(\kappa)$ - One to one mapping



$$y = \phi\left(\sum_{s=1}^{S} \theta_s u^{(s)} + \theta_0\right) = \phi\left(\theta^T \varphi(u)\right)$$

$$\theta = [\theta_0\ \theta_1 \cdots \theta_S]^T \qquad \varphi(u) = [1\ u^{(1)} \cdots u^{(S)}]^T$$

Uczenie: $[u_1\ u_2 \cdots u_N] = U_N \qquad [y_1\ y_2 \cdots y_N] = Y_N$

$$Q_N(\theta) = \sum_{n=1}^{N} (\kappa_n - \bar{\kappa}_n)^2 = \sum_{n=1}^{N} \left(\phi^{-1}(y_n) - \theta^T \varphi(u_n)\right)^2$$

Like before

$$\theta_N^* = \left[\sum_{n=1}^{N} \varphi(u_n)\varphi^T(u_n)\right]^{-1} \times \sum_{n=1}^{N} \phi^{-1}(y_n)\varphi(u_n)$$

Or recursive algorithm

Recursive algorithm

$$\theta^*_{N+1} = A\left(\theta^*_N, u_{N+1}, y_{N+1}\right) = \theta^*_N + K_{N+1}\left[\phi^{-1}\left(y_{N+1}\right) - \varphi\left(u_{N+1}\right)^T \theta^*_N\right]$$

$$K_{N+1} = \frac{P_N \varphi\left(u_{N+1}\right)}{1 + \varphi\left(u_{N+1}\right)^T P_N \varphi\left(u_{N+1}\right)}$$

$$P_{N+1} = P_N - \frac{P_N \varphi\left(u_{N+1}\right) \varphi\left(u_{N+1}\right)^T P_N}{1 + \varphi\left(u_{N+1}\right)^T P_N \varphi\left(u_{N+1}\right)}$$

Neuron learning – delta roll

$$Q(\theta) = \frac{1}{2}\left(y - \theta^T \varphi(u)\right)^2, \quad \theta_0$$

Numerical optimization method (it will be)

$$\theta^* \to Q\left(\theta^*\right) = \min_{\theta} Q(\theta)$$

$$\theta^*_{n+1} = \theta^*_n - \eta \nabla_{\theta} Q\left(\theta^*_n\right), \quad \theta^*_0$$

η – learning factor

$$\nabla Q(\theta) = -\left(y - \theta^T \varphi(u)\right)\varphi(u), \quad \theta_0$$

$$\theta^*_{n+1} = \theta^*_n + \eta\left(y_{n+1} - \theta^{*T}_n \varphi(u_{n+1})\right)\varphi(u_{n+1}), \quad \theta^*_0$$

$$\Delta_n = \left(y_{n+1} - \theta^{*T}_n \varphi(u_{n+1})\right)$$

$$\theta^*_{n+1} = \theta^*_n + \eta \Delta_n \varphi(u_{n+1})$$

Neuron learning – delta roll

$$Q(\theta)=\frac{1}{2}\left(y-\phi\left(\theta^{T}\varphi(u)\right)\right)^{2},\quad \theta_{0}$$

Φ – activation function, differentiable

Numerical optimization method (it will be)

$$\theta^{*}\rightarrow Q\left(\theta^{*}\right)=\min_{\theta} Q(\theta)$$

$$\theta_{n+1}^{*}=\theta_{n}^{*}-\eta\nabla_{\theta}Q\left(\theta_{n}^{*}\right),\ \ \theta_{0}^{*}$$

η - learning factor

$$\nabla Q(\theta)=-\left(y-\phi\left(\theta^{T}\varphi(u)\right)\right)\frac{\partial\phi(\kappa)}{\partial(\kappa)}\varphi(u),\quad \theta_{0}$$

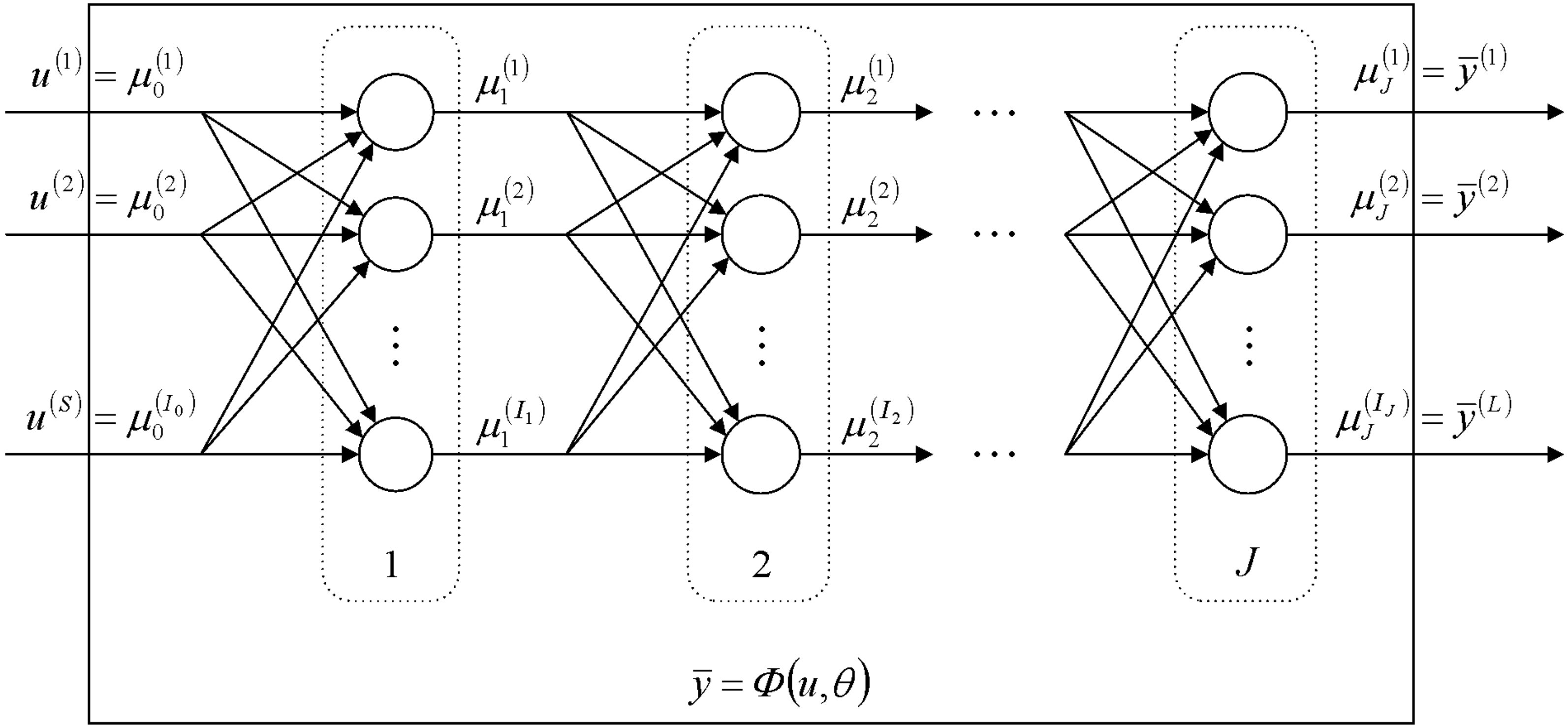
$$\theta_{n+1}^{*}=\theta_{n}^{*}+\eta\left(y_{n+1}-\phi\left(\theta_{n}^{*T}\varphi(u_{n+1})\right)\right)\frac{\partial\phi(\kappa)}{\partial\kappa}\varphi(u_{n+1}),\ \ \theta_{0}^{*}$$

$$\Delta_{n}=\left(y_{n+1}-\phi\left(\theta_{n}^{*T}\varphi(u_{n+1})\right)\right)\frac{\partial\phi(\kappa)}{\partial\kappa}$$

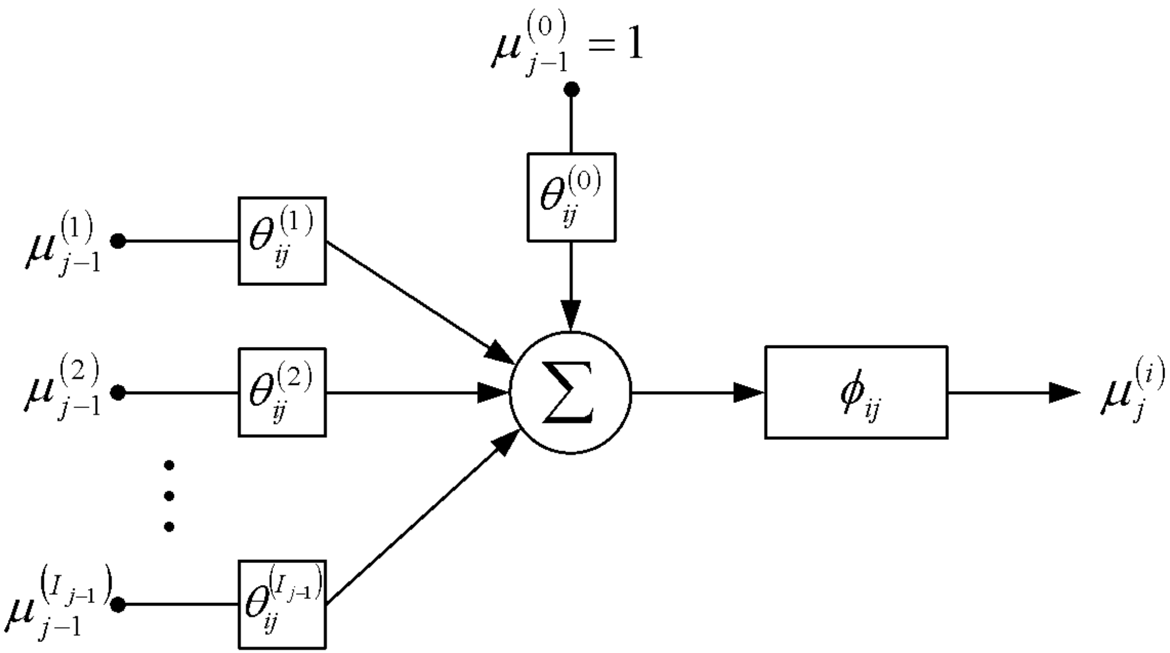
$$\theta_{n+1}^{*}=\theta_{n}^{*}+\eta\Delta_{n}\varphi(u_{n+1})$$

HUMAN CAPITAL
HUMAN – BEST INVESTMENT!
Wrocław University of Technology
EUROPEAN SOCIAL FUND
Project co-financed from the EU European Social Fund

Multilayer network



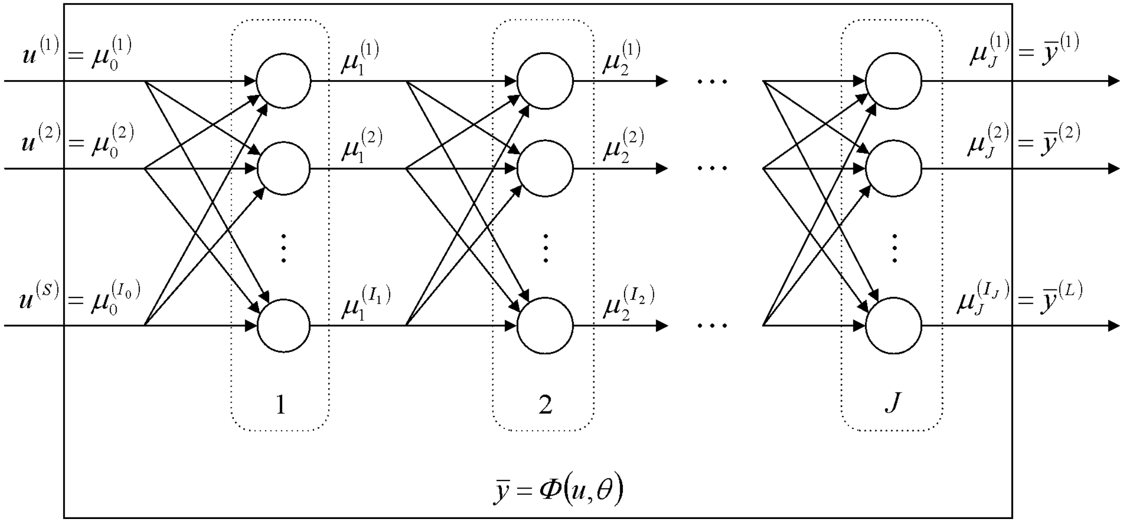
$\mu_j^{(i)} = \phi_{ij}\left(\sum_{s=1}^{I_{j-1}} \mu_{j-1}^{(s)}\theta_{ij}^{(s)} + \theta_{ij}^{(0)}\right),$



$$\mu_j^{(i)} = \phi_{ij}\left(\overline{\mu}_{j-1}^T \theta_{ij}\right).$$

$$\mu_j^{(i)} = \phi_{ij}\left(\sum_{s=1}^{I_{j-1}} \mu_{j-1}^{(s)}\theta_{ij}^{(s)} + \theta_{ij}^{(0)}\right), \quad \mu_{j-1} = \begin{bmatrix} \mu_{j-1}^{(1)} \\ \mu_{j-1}^{(2)} \\ \vdots \\ \mu_{j-1}^{(I_{j-1})} \end{bmatrix}, \quad \overline{\mu}_{j-1} \overset{\text{df}}{=} \begin{bmatrix} \mu_{j-1}^{(0)} \\ \mu_{j-1} \end{bmatrix} = \begin{bmatrix} \mu_{j-1}^{(0)} \\ \mu_{j-1}^{(1)} \\ \mu_{j-1}^{(2)} \\ \vdots \\ \mu_{j-1}^{(I_{j-1})} \end{bmatrix}, \quad \theta_{ij} \overset{\text{df}}{=} \begin{bmatrix} \theta_{ij}^{(0)} \\ \theta_{ij}^{(1)} \\ \vdots \\ \theta_{ij}^{(I_{j-1})} \end{bmatrix},$$

Multilayer network



$$\mu_j = \begin{bmatrix} \mu_j^{(1)} \\ \mu_j^{(2)} \\ \vdots \\ \mu_j^{(I_j)} \end{bmatrix} = \begin{bmatrix} \phi_{1j}\left(\overline{\mu}_{j-1}^T \theta_{1j}\right) \\ \phi_{2j}\left(\overline{\mu}_{j-1}^T \theta_{2j}\right) \\ \vdots \\ \phi_{I_j j}\left(\overline{\mu}_{j-1}^T \theta_{I_j j}\right) \end{bmatrix} \stackrel{\text{df}}{=} \phi_j\left(\overline{\mu}_{j-1}, \theta_j\right),$$

$$\theta_j \stackrel{\text{df}}{=} \begin{bmatrix} \theta_{1j} & \theta_{2j} & \cdots & \theta_{I_j j} \end{bmatrix} = \begin{bmatrix} \theta_{1j}^{(0)} & \theta_{2j}^{(0)} & \cdots & \theta_{I_j j}^{(0)} \\ \theta_{1j}^{(1)} & \theta_{2j}^{(1)} & \cdots & \theta_{I_j j}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{1j}^{(I_{j-1})} & \theta_{2j}^{(I_{j-1})} & \cdots & \theta_{I_j j}^{(I_{j-1})} \end{bmatrix}.$$

$$\mu_j = \phi_j\left(\overline{\mu}_{j-1}, \theta_j\right), \; j = 1, 2, \ldots, J.$$

$$\mu_1 = \phi_1\left(\overline{\mu}_0, \theta_1\right) = \phi_1\left(u, \theta_1\right).$$

$$\mu_j = \phi_j\left(\overline{\mu}_{j-1}, \theta_j\right), \; j = 2, 3, \ldots, J-1.$$

$$\overline{y} = \mu_J = \phi_J\left(\overline{\mu}_{J-1}, \theta_J\right).$$

$$\overline{y} = \begin{bmatrix} \overline{y}^{(1)} \\ \overline{y}^{(2)} \\ \vdots \\ \overline{y}^{(L)} \end{bmatrix} = \begin{bmatrix} \mu_J^{(1)} \\ \mu_J^{(2)} \\ \vdots \\ \mu_J^{(I_J)} \end{bmatrix} = \phi_J\left(\phi_{J-1}\left(\ldots \phi_1(u, \theta_1), \ldots, \theta_{J-1}\right), \theta_J\right) \stackrel{\text{df}}{=} \begin{bmatrix} \Phi^{(1)}(u, \theta) \\ \Phi^{(2)}(u, \theta) \\ \vdots \\ \Phi^{(L)}(u, \theta) \end{bmatrix} \stackrel{\text{df}}{=} \Phi(u, \theta),$$

$$\theta \stackrel{\text{df}}{=} \{\theta_1, \theta_2, \ldots, \theta_J\}.$$

Multilayer network learning

$$Q_n(\theta)=[y_n-\Phi(u_n,\theta)]^T[y_n-\Phi(u_n,\theta)]=\sum_{l=1}^{L}\left(y_n^{(l)}-\Phi^{(l)}(u_n,\theta)\right)^2=\sum_{l=1}^{L}\varepsilon_n^{(l)\,2}\qquad \varepsilon_n^{(l)}\overset{\text{df}}{=}y_n^{(l)}-\bar{y}_n^{(l)}=y_n^{(l)}-\Phi^{(l)}(u_n,\theta),\ l=1,2,\ldots,L$$

$$\tilde{\tilde{\theta}}_{ij,n+1}^{(s)}=\tilde{\tilde{\theta}}_{ij,n}^{(s)}-\eta_n\left.\frac{\partial Q_n(\theta)}{\partial\theta_{ij}^{(s)}}\right|_{\theta_{ij}^{(s)}=\tilde{\tilde{\theta}}_{ij,n}^{(s)}},\ s=0,1,\ldots,I_{J-1},\ i=1,2,\ldots,I_J,\ j=1,2,\ldots,J,$$

$$\left.\frac{\partial Q_n(\theta)}{\partial\theta_{ij}^{(s)}}\right|_{\theta_{ij}^{(s)}=\tilde{\tilde{\theta}}_{ij,n}^{(s)}}=-2\delta_{jn}^{(i)}\mu_{j-1,n}^{(s)},$$

$$\varepsilon_{jn}^{(i)}\overset{\text{df}}{=}\begin{cases}\varepsilon_n^{(i)} & \text{dla}\ \ j=J\\ \sum_{p=1}^{I_{j+1}}\delta_{j+1,n}^{(p)}\tilde{\tilde{\theta}}_{pj+1,n}^{(i)} & \text{dla}\ \ j=J-1,\ J-2,\ldots,1\end{cases}$$

$$\delta_{jn}^{(i)}\overset{\text{df}}{=}\varepsilon_{jn}^{(i)}\left.\frac{d\phi_{ij}(\kappa_j^{(i)})}{d\kappa_j^{(i)}}\right|_{\theta_{ij}^{(s)}=\tilde{\tilde{\theta}}_{ij,n}^{(s)}}$$

$$\kappa_j^{(i)}\overset{\text{df}}{=}\sum_{s=1}^{I_{j-1}}\mu_{j-1}^{(s)}\theta_{ij}^{(s)}+\theta_{ij}^{(0)}.$$

$$\tilde{\tilde{\theta}}_{ij,n+1}^{(s)}=\tilde{\tilde{\theta}}_{ij,n}^{(s)}+2\eta_n\delta_{jn}^{(i)}\mu_{j-1,n}^{(s)},\ \ s=0,1,\ldots,I_{J-1},\ \ i=1,2,\ldots,I_J,\ \ j=1,2,\ldots,J.$$

Thank you for attention