Computer Science

Jerzy Świątek

Systems Modelling and Analysis

L.3b. Estimation of plant parameter with noisy measurements

IDENTIFICATION
PLANT

Known plant
characteristics

PLANT IN THE
CLASS
OF MODELS

Unknown plant
characteristics

CHOICE
OF THE
BEST MODEL

Exact measurements

DETERMINATION
OF THE PLANT
PARAMETERS

Noised measurements

PARAMETER
ESTIMATION
OF THE PLANT
CHARACTERISTIC

Deterministic plant

CHOICE
OF THE
BEST MODEL
DETERMINISTIC
PROBLEM

Random plant

CHOICE
OF THE
BEST MODEL
PROBABILISTIC
PROBLEM

Plant in the class of model

Choice of the best model

Deterministyczny

$$u_n \rightarrow \boxed{F(u,\theta)} \rightarrow y_n$$

$$U_N = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix},$$
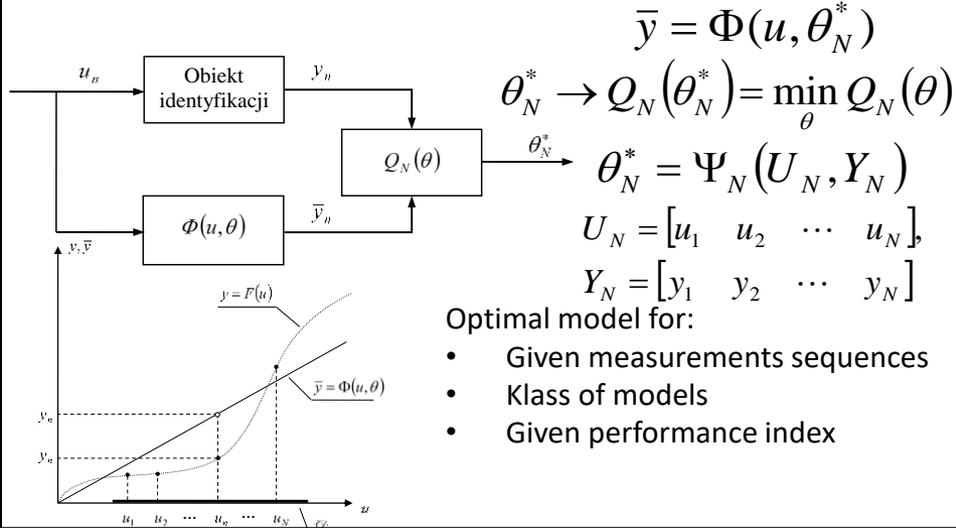$$Y_N = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}$$
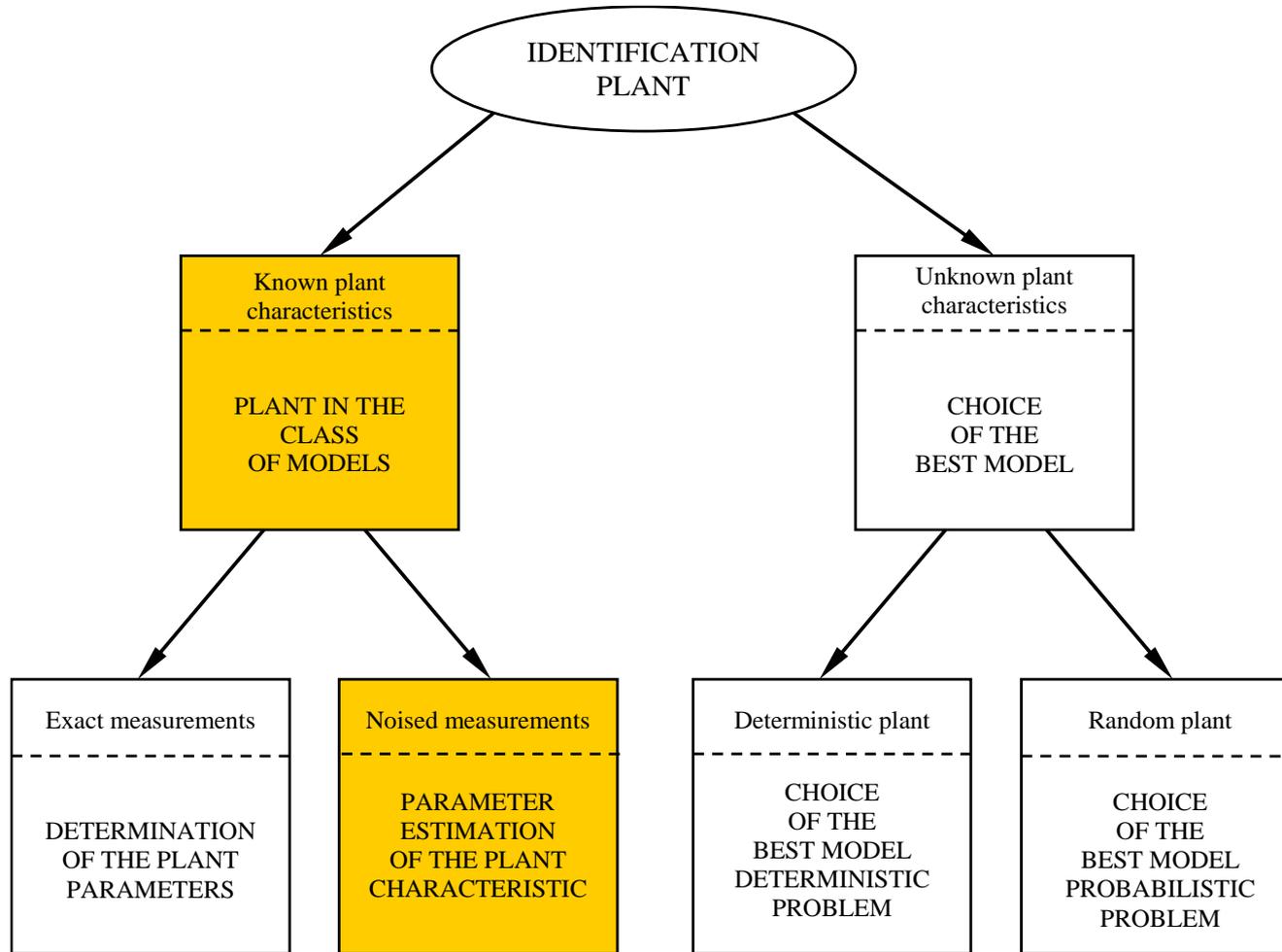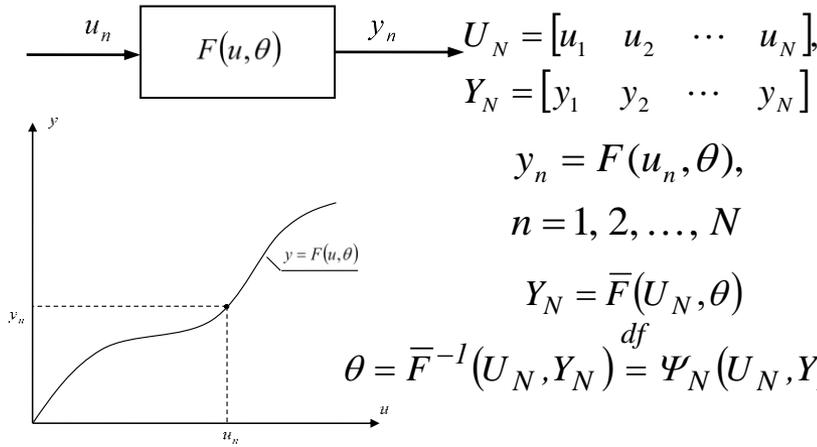
$$y_n = F(u_n,\theta),$$
$$n = 1, 2, \ldots, N$$

$$Y_N = \overline{F}(U_N,\theta)$$

$$\theta = \overline{F}^{-1}(U_N,Y_N) \overset{df}{=} \Psi_N(U_N,Y_N)$$

$y = F(u,\theta)$

$$u_n \rightarrow \boxed{\begin{array}{c}\text{Obiekt}\\\text{identyfikacji}\end{array}} \rightarrow y_n$$

$$\boxed{Q_N(\theta)} \overset{\theta_N^*}{\rightarrow}$$

$$\boxed{\Phi(u,\theta)} \rightarrow \overline{y}_n$$

$$\overline{y} = \Phi(u,\theta_N^*)$$
$$\theta_N^* \rightarrow Q_N(\theta_N^*) = \min_\theta Q_N(\theta)$$
$$\theta_N^* = \Psi_N(U_N,Y_N)$$

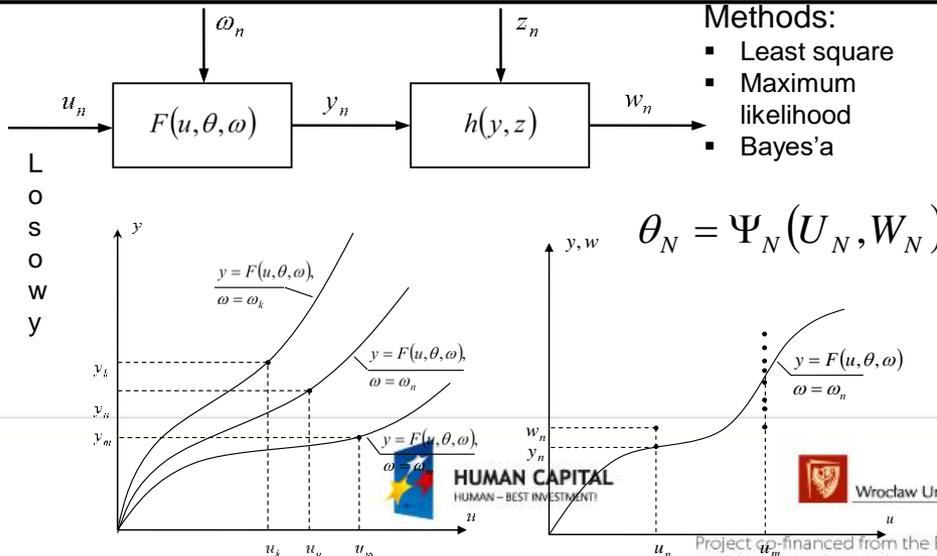$$U_N = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix},$$
$$Y_N = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}$$

Optimal model for:
- Given measurements sequences
- Klass of models
- Given performance index

$y = F(u)$
$\overline{y} = \Phi(u,\theta)$

Losowy

$$u_n \rightarrow \boxed{F(u,\theta,\omega)} \overset{y_n}{\rightarrow} \boxed{h(y,z)} \rightarrow w_n$$

with ω_n and z_n inputs.

Methods:
- Least square
- Maximum likelihood
- Bayes'a

$$\theta_N = \Psi_N(U_N,W_N)$$

$y = F(u,\theta,\omega),\ \omega = \omega_k$
$y = F(u,\theta,\omega),\ \omega = \omega_n$
$y = F(u,\theta,\omega),\ \omega = \omega_n$
$y = F(u,\theta,\omega),\ \omega = \omega_n$

$$\overline{y} = \Phi^*(u) = E[\underline{y}|\underline{u} = u]$$
$$\overline{y} = \Phi(u,\theta^*)$$
$$E[\underline{y}|\underline{u} = u']$$
$$f_y(y|u')$$
$$f_u(u)$$

$$f_y(y|u)$$
$$f_u(u)$$
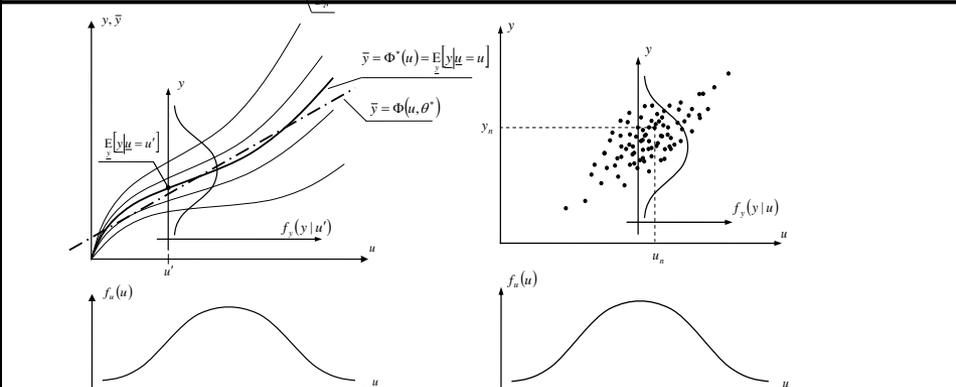
Full probabilistic knowledge
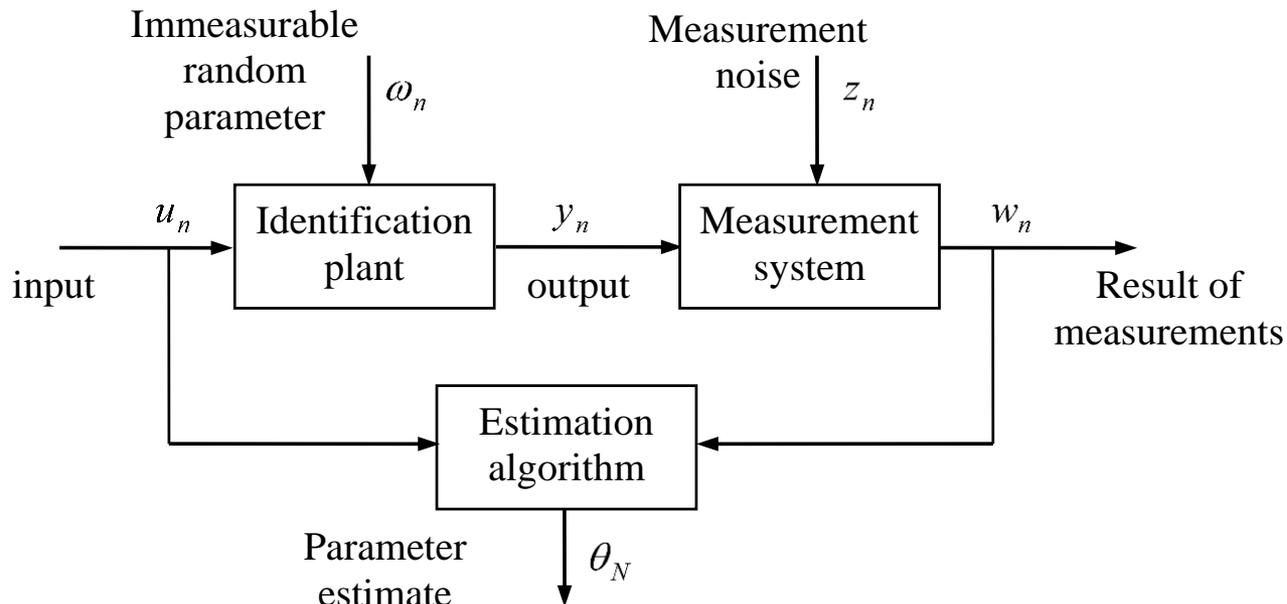- First type regrsesion
- Second type regression

Unknown probabilostic knowledge
- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

Plant parameter estimation problem

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Noised measurements of the physical values

- ## Problem formulation

Measurement noise:

z_n – value of random variable $\underline{z}$ from the space $\mathscr{X}$

$f_z(z)$ – probability density function

θ – observed vector of parameters, value of random variable $\underline{\theta}, \ \theta \in \Theta \subseteq \mathscr{R}^R$

$f_\theta(\theta)$ – probability density function

Measurements: $V_N = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$

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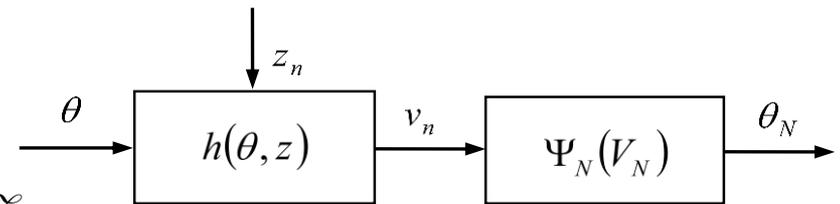
Noised measurements of the physical values

General form of estimation algorithm:

$$\theta_N = \Psi_N(V_N)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayesian method

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Deterministic plant, noised measurements of the plant output

- Noised measurements of the identification plant known static characteristics
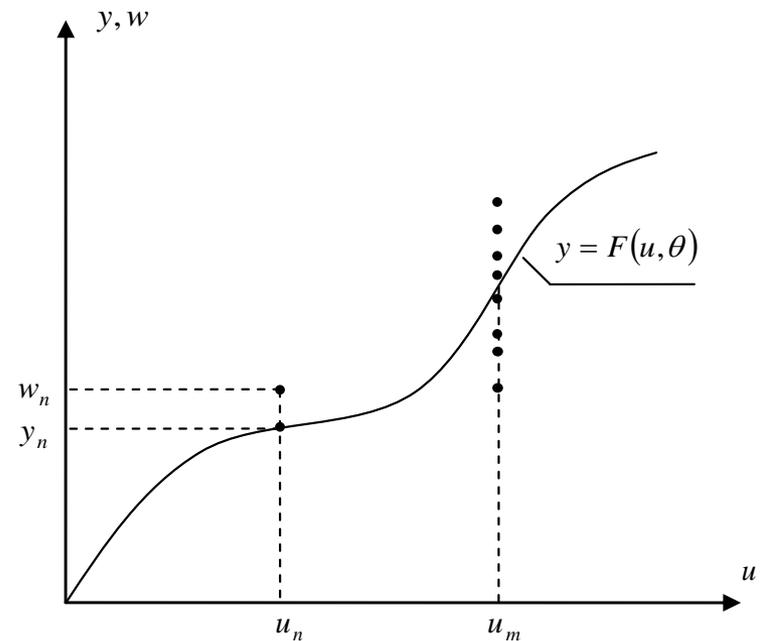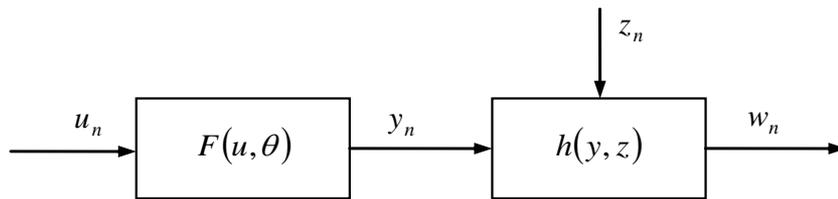
Deterministic plant, noised measurements of the plant output

- Problem formulation

Measurement system description: $\quad w = h(y, z)$

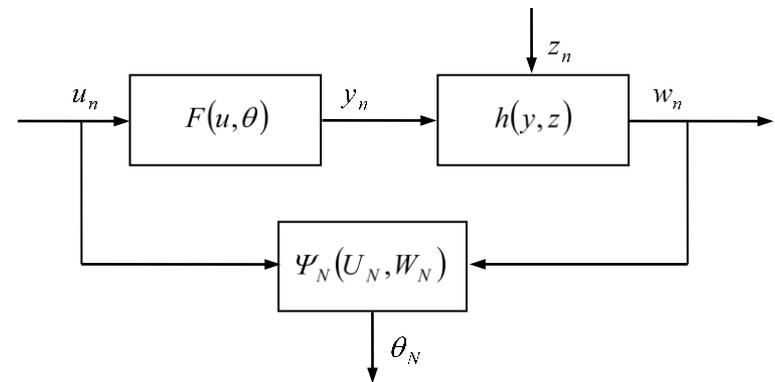where: $\quad w \in \mathscr{W}, \quad h$ – known one-to-one function

$$h : \mathscr{Y} \times \mathscr{Z} \to \mathscr{W}, \quad z = h_z^{-1}(y, w)$$
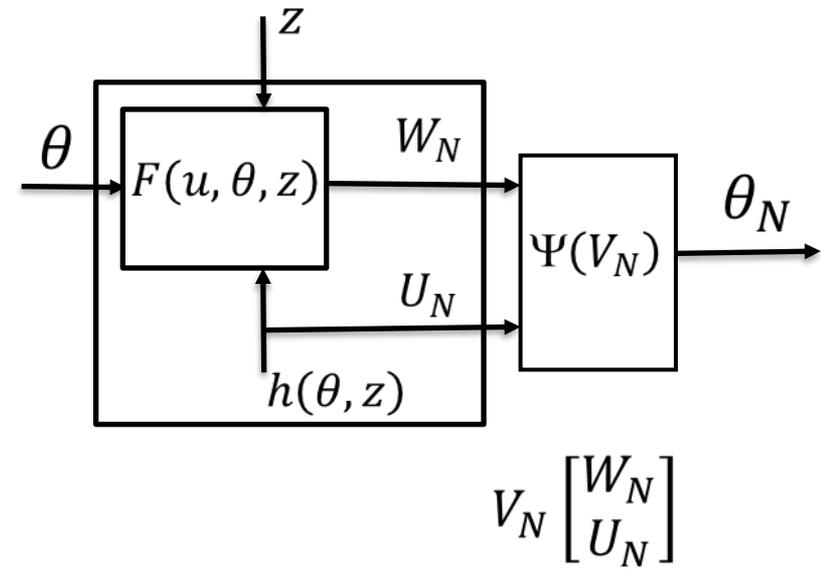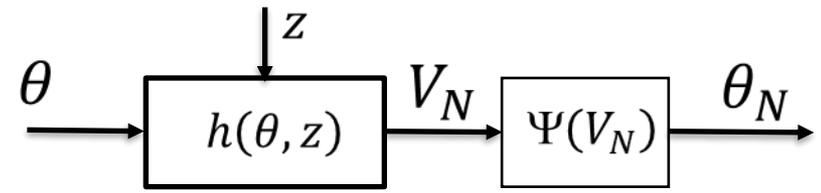
$\mathscr{W}$ – measurements domain ($\dim y = \dim z = L$)

Measurement noise:

z_n – value of random variable $\underline{z}$ from the space $\mathscr{Z}$ $\qquad f_z(z)$ – probability density function

Measurements: $\quad U_N = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}, \; W_N = \begin{bmatrix} w_1 & w_2 & \cdots & w_N \end{bmatrix}$

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Deterministic plant, noised measurements of the plant output

General form of estimation algorithm:

$$\theta_N = \Psi_N\left(U_N, W_N\right)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayes' method

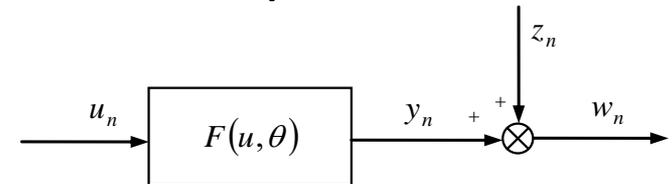HUMAN CAPITAL
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Least square method

(one dimensional case $L = 1$)

Assumptions:

$$w = h(y, z) = y + z \quad \text{– additive noise}$$

$$E_{\underline{z}}[\underline{z}] = 0 \quad \text{– expected value of the noise signal is zero}$$

$$Var_{\underline{z}}[\underline{z}] < \infty \quad \text{– variance of noise is not infinite}$$

Calculations:

Least square method minimizes empirical variance of noise signal:

$$Var_{zN}(U_N, W_N, \theta) = \frac{1}{N}\sum_{n=1}^{N}(w_n - y_n)^2 = \frac{1}{N}\sum_{n=1}^{N}(w_n - F(u_n, \theta))^2$$
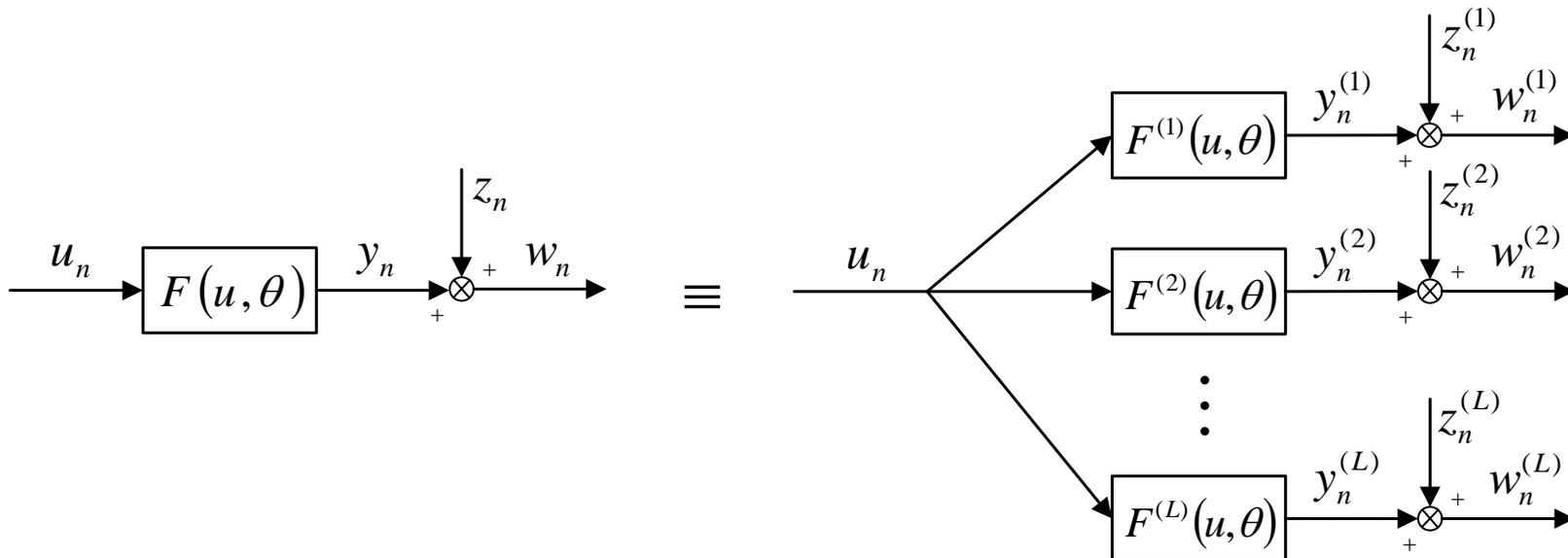
Estimation algorithm has the form:

$$\theta_N = \Psi_N(U_N, W_N) \quad \rightarrow \quad Var_{zN}(U_N, W_N, \theta_N) = \min_{\theta \in \Theta} Var_{zN}(U_N, W_N, \theta)$$
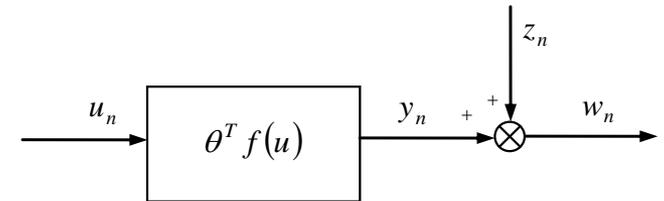
Least square method

Decomposition of L – dimensional identification plant into L single output plants:

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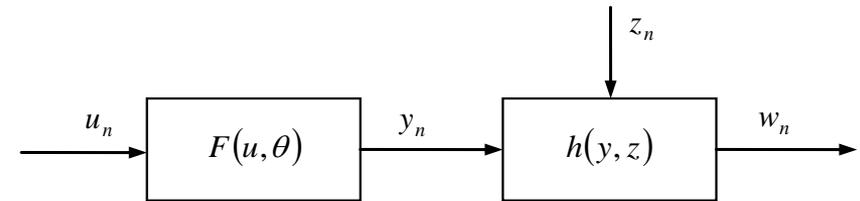# Least square method

- Example

Empirical variance of noise signal:

$$Var_{zN}(U_N, W_N, \theta) = \frac{1}{N}\sum_{n=1}^{N}(w_n - y_n)^2 = \frac{1}{N}\sum_{n=1}^{N}(w_n - \theta^T f(u_n))^2$$

Estimation algorithm:

$$\theta_N = \Psi_N(U_N, W_N) = \left[\sum_{n=1}^{N} f(u_n) f^T(u_n)\right]^{-1} \sum_{n=1}^{N} w_n f(u_n)$$

Maximum likelihood method

Assumptions:

$\underline{w} = h(y, \underline{z})$ – measurement system is described by any one-to-one invertible function

z_n – independent value of random variable $\underline{z}$ with known probability density function $f_z(z)$

For a given $u_n, n = 1, 2, \ldots, N$ noised output is measured $w_n, n = 1, 2, \ldots, N$
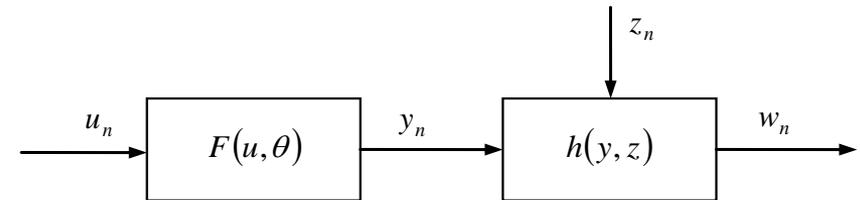
Calculations:

Taking into account plant description $\underline{w} = h(F(u, \theta), \underline{z})$, probability density function of random variable $\underline{w} - f_w(w, \theta; u)$ with parameters θ has the form: $f_w(w, \theta; u) = f_z\left(h_z^{-1}(F(u, \theta), w)\right) \cdot |J_h|$

where J_h is Jacobi matrix: $J_h = \dfrac{\partial h_z^{-1}(y, w)}{\partial w}$

Maximum likelihood method

Likelihood function:

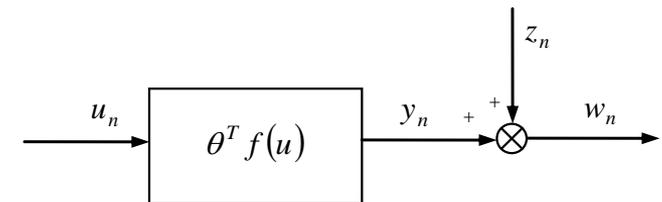$$L_N(W_N,\theta;U_N) = \prod_{n=1}^{N} f_w(w_n,\theta;u_n) = \prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n,\theta),w_n)\right)|J_h|$$

Estimation algorithm:

$$\theta_N = \Psi_N(U_N,W_N) \quad \rightarrow \quad L_N(W_N,\theta_N;U_N) = \max_{\theta\in\Theta} L_N(W_N,\theta;U_N)$$

Maximum likelihood method

- Example 1

Noise description:
$$f_z(z) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\frac{(z - m_z)^2}{2\sigma_z^2} \right]$$

Measurement system description: $w = h(y, z) = y + z$

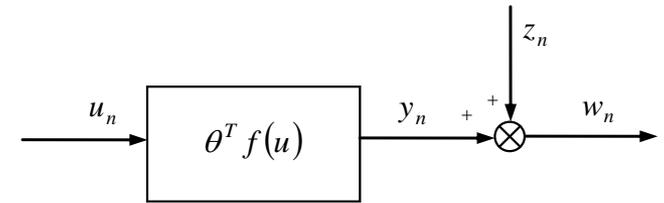$$z = h_z^{-1}(F(u, \theta), w) = w - \theta^T f(u)$$

Jacobi matrix of the inverse transform :
$$J_h = \frac{\partial h_z^{-1}(F(u, \theta), w)}{\partial w} = \frac{d}{dw}\left(w - \theta^T f(u) \right) = 1$$

Maximum likelihood method

- Example 1

Probability density function : $f_w(w, \theta; u) = \dfrac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\dfrac{\left(w - \theta^T f(u) - m_z\right)^2}{2\sigma_z^2} \right] \cdot |1|$

Likelihood function : $L_N(W_N, \theta; U_N) = \prod_{n=1}^{N} \dfrac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\dfrac{\left(w_n - \theta^T f(u_n) - m_z\right)^2}{2\sigma_z^2} \right]$

$$L_N(W_N, \theta; U_N) = \left(\dfrac{1}{\sigma_z \sqrt{2\pi}} \right)^N \exp\left[\sum_{n=1}^{N} -\dfrac{\left(w_n - \theta^T f(u_n) - m_z\right)^2}{2\sigma_z^2} \right]$$
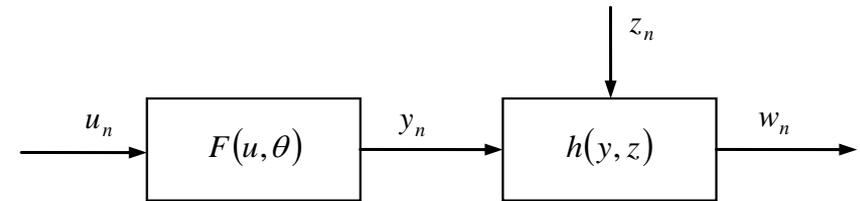
Estimation algorithm:
$$\theta_N = \Psi_N(U_N, W_N) = \left[\sum_{n=1}^{N} f(u_n) f^T(u_n) \right]^{-1} \sum_{n=1}^{N} (w_n - m_z) f(u_n)$$

Maximum likelihood method

- Example 2

Plant characteristic: $\quad y = F(u, \theta) = \theta\, u \qquad (\theta > 0)$

Noise description: $\quad f_z(z) = \begin{cases} 1 & \text{for} \quad z \in [0, 1] \\ 0 & \text{for} \quad z \notin [0, 1] \end{cases}$

Additional assumption: $\quad \forall\, n = 1, 2, \ldots, N \quad u_n > 0$

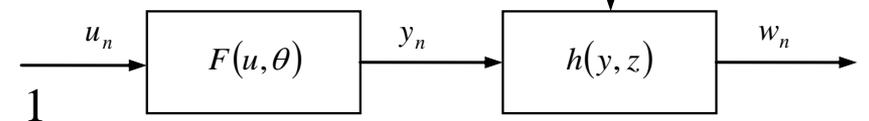Measurement system description: $\quad w = h(y, z) = h(F(u, \theta), z) = \theta\, u\, z$

$$z = h_z^{-1}(F(u, \theta), w) = \frac{w}{\theta\, u}$$
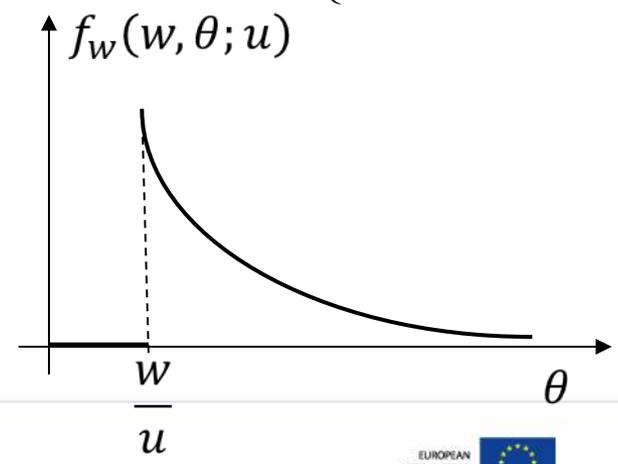
Maximum likelihood method

- Example 2

$$u_n \rightarrow \boxed{F(u,\theta)} \xrightarrow{y_n} \boxed{h(y,z)} \xrightarrow{w_n}$$

with z_n entering $h(y,z)$.

Jacobi matrix: $\quad J_h = \dfrac{\partial h_z^{-1}\big(F(u,\theta),w\big)}{\partial w} = \dfrac{d}{dw}\left(\dfrac{w}{\theta u}\right) = \dfrac{1}{\theta u}$

Probability density function of the observed value: $\quad f_w(w,\theta;u) = \begin{cases} \dfrac{1}{\theta u} & \text{for} \quad \dfrac{w}{\theta u} \in [0,1] \\[2mm] 0 & \text{for} \quad \dfrac{w}{\theta u} \notin [0,1] \end{cases}$

$$f_w(w,\theta;u) = \begin{cases} \dfrac{1}{\theta}u & for \ \dfrac{w}{u} \geq \theta \\[2mm] 0 & for \ \dfrac{w}{u} < \theta \end{cases}$$

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Maximum likelihood method

- Example 2

Likelihood function :

$$L_N\left(W_N,\theta;U_N\right)=\begin{cases}\dfrac{1}{\theta\displaystyle\prod_{n=1}^{N}u_n} & \text{for}\quad \forall n=1,2,\ldots,N\quad \dfrac{w_n}{u_n}\in[0,\theta]\\[20pt]0 & \text{for}\quad \exists n=1,2,\ldots,N\quad \dfrac{w_n}{u_n}\notin[0,\theta]\end{cases}$$

$$L_N\left(W_N,\theta;U_N\right)=\begin{cases}\dfrac{1}{\theta\displaystyle\prod_{n=1}^{N}u_n} & \text{for}\quad \theta\geq\max_{1\leq n\leq N}\left\{\dfrac{w_n}{u_n}\right\}\\[20pt]0 & \text{for}\quad \theta<\max_{1\leq n\leq N}\left\{\dfrac{w_n}{u_n}\right\}\end{cases}$$

Maximum likelihood method

- Example 2

$$u_n \longrightarrow \boxed{F(u,\theta)} \xrightarrow{y_n} \boxed{h(y,z)} \xrightarrow{w_n}$$

with z_n input to $h(y,z)$
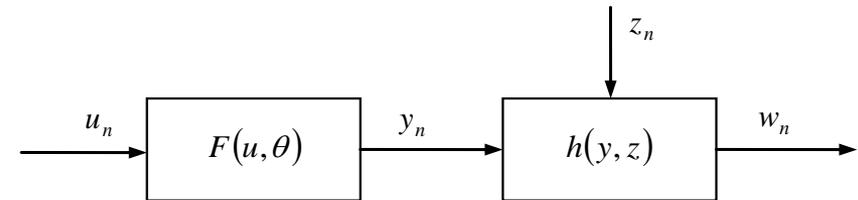
Likelihood function :

$$L_N\left(W_N,\theta;U_N\right) = \begin{cases} \dfrac{1}{\theta \displaystyle\prod_{n=1}^{N} u_n} & \text{for} \quad \forall n = 1,2,\ldots,N \quad \dfrac{w_n}{u_n} \in [0,\theta] \\[6mm] 0 & \text{for} \quad \exists n = 1,2,\ldots,N \quad \dfrac{w_n}{u_n} \notin [0,\theta] \end{cases}$$

$$L_N\left(W_N,\theta;U_N\right) = \begin{cases} \dfrac{1}{\theta \displaystyle\prod_{n=1}^{N} u_n} & \text{for} \quad \theta \geq \max_{1\leq n\leq N}\left\{\dfrac{w_n}{u_n}\right\} \\[6mm] 0 & \text{for} \quad \theta < \max_{1\leq n\leq N}\left\{\dfrac{w_n}{u_n}\right\} \end{cases}$$

Maximum likelihood method

- Example 2
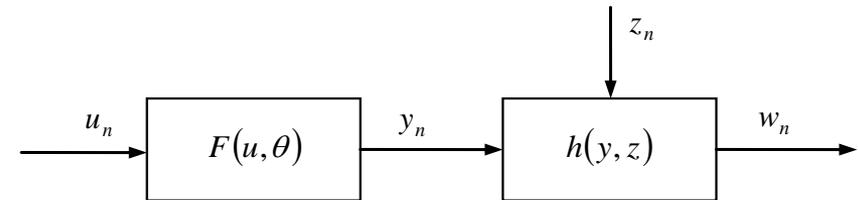
Estimation algorithm:

$$\theta_N = \Psi_N(U_N, W_N) = \max_{1 \le n \le N}\left\{\frac{w_n}{u_n}\right\}$$

Interpretation:

$$\theta_N = \max_{1 \le n \le N}\left\{\frac{w_n}{u_n}\right\} = \max_{1 \le n \le N}\left\{\frac{\theta\, u_n\, z_n}{u_n}\right\} = \theta \max_{1 \le n \le N}\left\{z_n\right\}$$
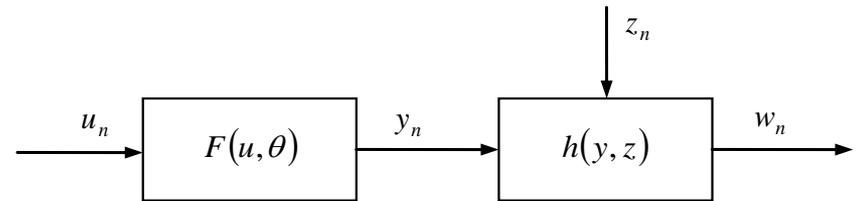
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Bayesian method's

Assumptions:

$\underline{w} = h(y, \underline{z})$ – measurement system is described by any one-to-one invertible function

z_n – independent value of random variable $\underline{z}$ with known probability density function $f_z(z)$

Mathematical formulas describing probability density functions $f_z(z)$ and $f_\theta(\theta)$ are given.

The loss function $L(\theta, \bar{\theta})$ is defined.
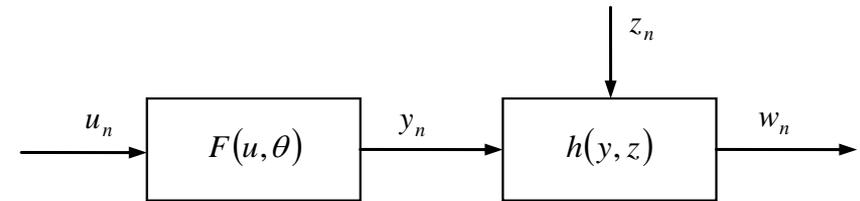
For a given $u_n, n = 1, 2, \ldots, N$ noised output is measured $w_n, n = 1, 2, \ldots, N$

θ – plant parameters vector, value of random variable $\underline{\theta}, \ \theta \in \Theta \subseteq \mathcal{R}^R$

Bayesian method's

Calculations:

Taking into account plant description: $\underline{w} = h(F(u, \theta), \underline{z})$

Risk: $R(\overline{\Psi}) \overset{\text{df}}{=} \underset{\theta, \underline{W_N}}{E} \left[L\left(\underline{\theta}, \overline{\underline{\theta}} = \overline{\Psi}(U_N, \underline{W_N})\right)\right] = \int\limits_{\mathscr{W}_N} \int\limits_{\Theta} L\left(\theta, \overline{\Psi}(U_N, W_N)\right) f(\theta, W_N; U_N) \, d\theta \, dW_N$

where $f(\theta, W_N; U_N)$ is joint probability density function:

$$f(\theta, W_N; U_N) = f'(\theta | W_N; U_N) f''(W_N; U_N)$$

The problem: $\Psi_N \rightarrow R(\Psi_N) = \min_{\overline{\Psi}} R(\overline{\Psi})$

Bayesian method's

Joint probability density function: $f(\theta, W_N; U_N) = f'(\theta|W_N; U_N) f''(W_N; U_N)$

where: $f'(\theta|W_N; U_N) = \dfrac{f_\theta(\theta) f_{WN}(W_N|\theta; U_N)}{f''(W_N; U_N)} = \dfrac{f_\theta(\theta) f_{WN}(W_N|\theta; U_N)}{\displaystyle\int_\Theta f_\theta(\theta) f_{WN}(W_N|\theta; U_N) d\theta}$

$$f_{WN}(W_N|\theta; U_N) = \prod_{n=1}^{N} f_w(w_n|\theta; u_n)$$

Conditional probability density function of observed value: $\quad f_w(w|\theta; u) = f_z\left(h_z^{-1}(F(u, \theta), w)\right)|J_h|$

$$f'(\theta|W_N; U_N) = \frac{f_\theta(\theta) \displaystyle\prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n, \theta), w_n)\right)|J_h|}{\displaystyle\int_\Theta f_\theta(\theta) \prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n, \theta), w_n)\right)|J_h| d\theta} \quad \alpha \quad f_\theta(\theta) \prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n, \theta), w_n)\right)|J_h|$$

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Bayesian method's

Risk:

$$R(\overline{\Psi}) = \int\limits_{\mathcal{W}_N} \int\limits_{\Theta} L(\theta, \overline{\Psi}(U_N, W_N)) f'(\theta | W_N; U_N) d\theta \, f''(W_N; U_N) \, dW_N$$

Conditional risk:

$$r(\overline{\theta}, W_N; U_N) \stackrel{\mathrm{df}}{=} \underset{\underline{\theta}}{E} \left[L(\underline{\theta}, \overline{\theta}) | W_N; U_N \right] = \int\limits_{\Theta} L(\theta, \overline{\theta} = \overline{\Psi}(U_N, W_N)) f'(\theta | W_N; U_N) d\theta$$

Estimation algorithm:

$$\theta_N = \Psi_N(U_N, W_N) \quad \rightarrow \quad r(\theta_N, W_N; U_N) = \min_{\overline{\theta} \in \Theta} r(\overline{\theta}, W_N; U_N)$$

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Bayesian method's

- ## Mean a posteriori value method

Special case of the loss function: $L(\theta, \bar{\theta}) = [\theta - \bar{\theta}]^T [\theta - \bar{\theta}]$

Conditional risk: $r(\bar{\theta}, W_N; U_N) \overset{df}{=} \underset{\underline{\theta}}{E} \left[[\underline{\theta} - \bar{\theta}]^T [\underline{\theta} - \bar{\theta}] \big| W_N; U_N \right] =$

$$= \underset{\underline{\theta}}{E} \left[\underline{\theta}^T \underline{\theta} \big| W_N; U_N \right] - 2\bar{\theta} \underset{\underline{\theta}}{E} \left[\underline{\theta}^T \big| W_N; U_N \right] + \bar{\theta}^T \bar{\theta}$$

$$\underset{\bar{\theta}}{\mathrm{grad}}\, r(\bar{\theta}, W_N; U_N) \Big|_{\bar{\theta} = \theta_N} = -2 \underset{\underline{\theta}}{E} \left[\underline{\theta} \big| W_N; U_N \right] + 2\theta_N = 0_N$$

Estimation algorithm:

$$\theta_N = \Psi_N(U_N, W_N) = \underset{\underline{\theta}}{E} \left[\underline{\theta} \big| W_N; U_n \right] = \int_{\Theta} \theta f'(\theta | W_N; U_N) d\theta$$

Bayesian method's

- Maximum a posteriori probability

Special case of the loss function: $L(\theta, \bar{\theta}) = -\delta(\theta - \bar{\theta})$

Conditional risk: $r(\bar{\theta}, W_N; U_N) = -\int_\Theta \delta(\theta - \bar{\theta}) f'(\theta | W_N; U_N) d\theta = -f'(\bar{\theta} | W_N; U_N)$

$$\theta_N = \Psi_N(U_N, W_N) \quad \rightarrow \quad f'(\theta_N | W_N; U_N) = \max_{\bar{\theta} \in \Theta} f'(\bar{\theta} | W_N; U_N)$$

$$f'(\theta_N | W_N; U_N) \quad \alpha \quad f_\theta(\theta_N) \prod_{n=1}^N f_z\left(h_z^{-1}(F(u_n, \theta_N), w_n)\right) |J_h|$$
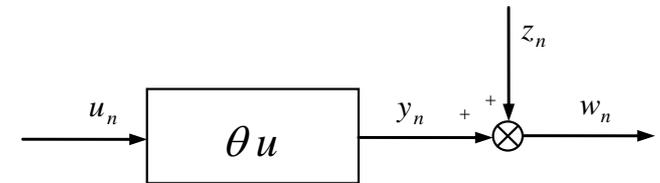
$\theta_N = \Psi_N(U_N, W_N)$ such that:

$$f_\theta(\theta_N) \prod_{n=1}^N f_z\left(h_z^{-1}(F(u_n, \theta_N), w_n)\right) |J_h| = \max_{\bar{\theta} \in \Theta} f_\theta(\bar{\theta}) \prod_{n=1}^N f_z\left(h_z^{-1}(F(u_n, \bar{\theta}), w_n)\right) |J_h|$$

Bayesian method's

- ## Example

Plant characteristic: $\quad y = F(u, \theta) = \theta\, u$

Noise description: $\quad f_z(z) = \dfrac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\dfrac{z^2}{2\sigma_z^2} \right]$
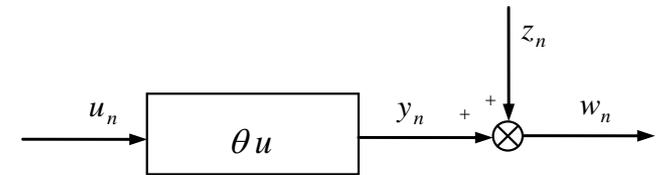
A priori distribution: $\quad f_\theta(\theta) = \dfrac{1}{\sigma_\theta \sqrt{2\pi}} \exp\left[-\dfrac{(\theta - m_\theta)^2}{2\sigma_\theta^2} \right]$

Measurement system description: $\quad w = h(y, z) = y + z, \quad z = h_z^{-1}(F(u, \theta), w) = w - \theta\, u$

Bayesian method's

- Example

Jacobi matrix:

$$J_h = \frac{\partial h_z^{-1}\left(F(u,\theta),w\right)}{\partial w} = \frac{d}{dw}\left(w - \theta\,u\right) = 1$$

Probability density function of the observed value: $f_w\left(w|\theta;u\right) = \frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{\left(w-\theta\,u\right)^2}{2\sigma_z^2}\right]\cdot\left|1\right|$

A posteriori probability density function:

$$f'\left(\theta|W_N;U_N\right) \quad \alpha \quad f_\theta\left(\bar\theta\right)\prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F(u_n,\theta),w_n\right)\right)\left|J_h\right| =$$

$$= \frac{1}{\sigma_\theta\sqrt{2\pi}}\exp\left[-\frac{\left(\bar\theta - m_\theta\right)^2}{2\sigma_\theta^2}\right]\prod_{n=1}^{N}\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{\left(w_n - \bar\theta\,u_n\right)^2}{2\sigma_z^2}\right]$$

Bayesian method's
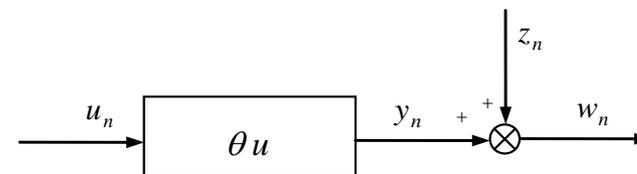
- Example

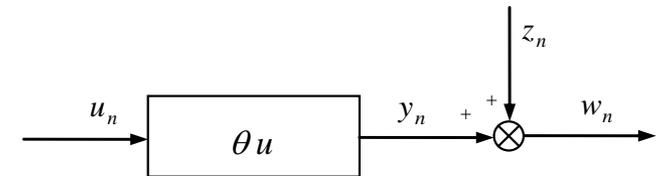A posteriori probability density function after transformation:

$$f'\left(\overline{\theta}\big|V_N\right) \quad \alpha \quad f_\theta\left(\overline{\theta}\right) \prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F\left(u_n,\overline{\theta}\right),w_n\right)\right)\left|J_h\right| =$$

$$= \frac{1}{\sigma_\theta\sqrt{2\pi}}\left(\frac{1}{\sigma_z\sqrt{2\pi}}\right)^N \exp\left[-\frac{\left(\overline{\theta}-m_\theta\right)^2}{2\sigma_\theta^2} - \sum_{n=1}^{N}\frac{\left(w_n-\overline{\theta}\,u_n\right)^2}{2\sigma_z^2}\right]$$

Bayesian method's

- ## Example

Estimation algorithm:

$$\theta_N = \Psi_N\big(U_N, W_N\big) = \frac{m_\theta + \left(\dfrac{\sigma_\theta}{\sigma_z}\right)^2 \sum\limits_{n=1}^{N} w_n u_n}{1 + \left(\dfrac{\sigma_\theta}{\sigma_z}\right)^2 \sum\limits_{n=1}^{N} u_n^2}$$

Discussion:

1° N – small number

$\big(\sigma_z \gg \sigma_\theta\big)$ – poor measurements

$$\theta_N \approx m_\theta$$

2° $N \to \infty$

$\big(\sigma_z \ll \sigma_\theta\big)$ – good measurements

$$\theta_N \approx \frac{\sum\limits_{n=1}^{N} w_n u_n}{\sum\limits_{n=1}^{N} u_n^2}$$

Thank you for attention

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