Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.3c. Estimation of the plant parameter with random value

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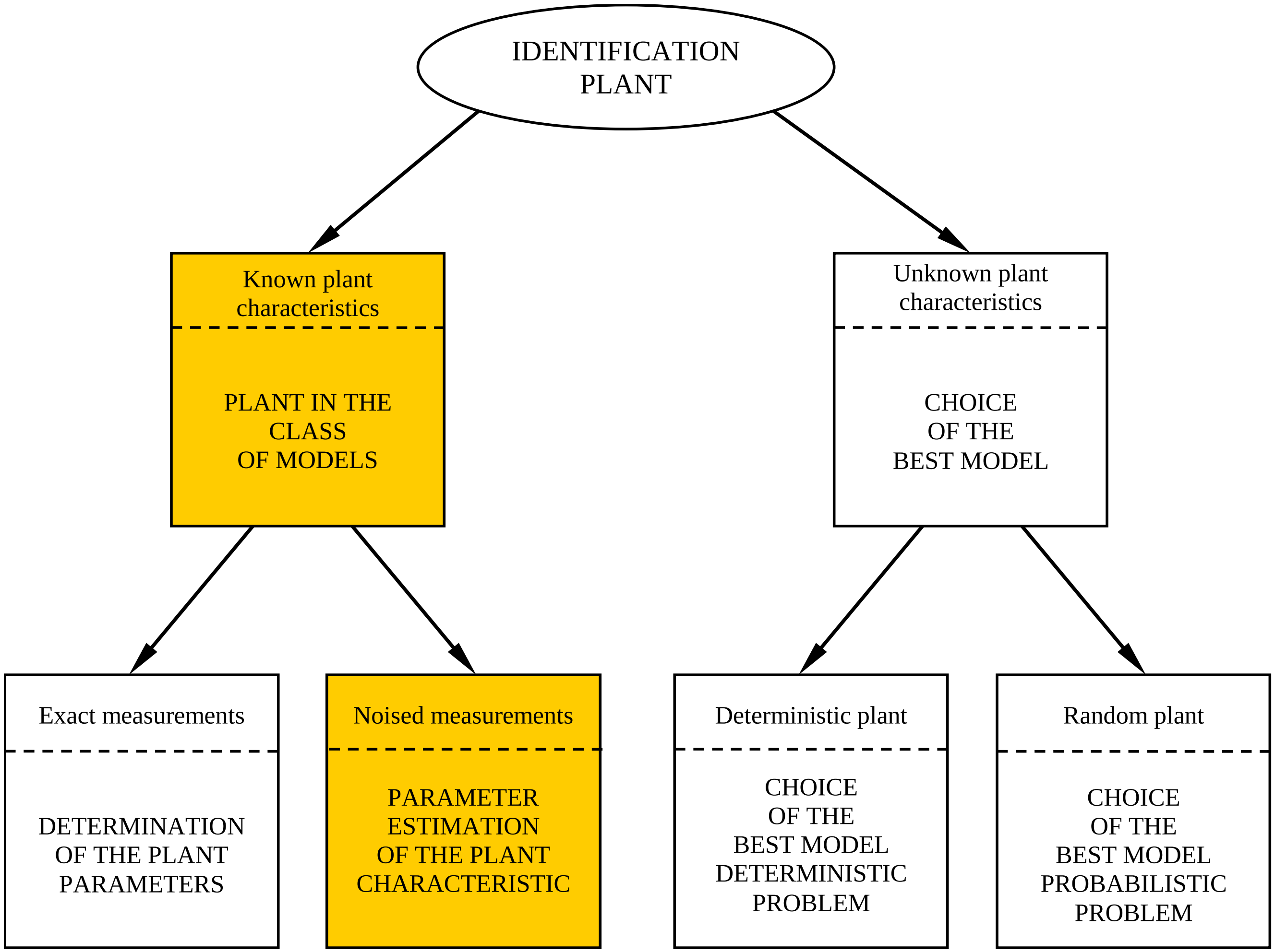

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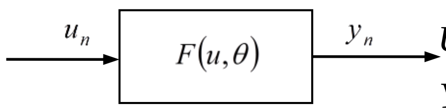
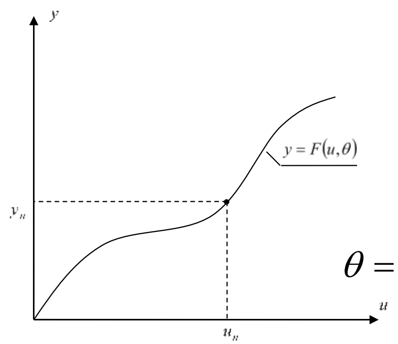
Plant in the class of model

D e t e r m i n i s t y c z n y

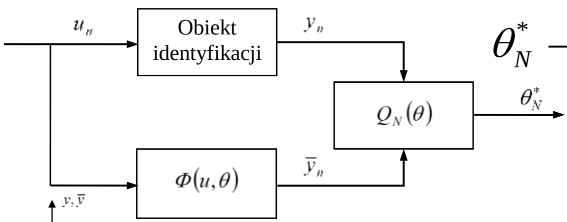


$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

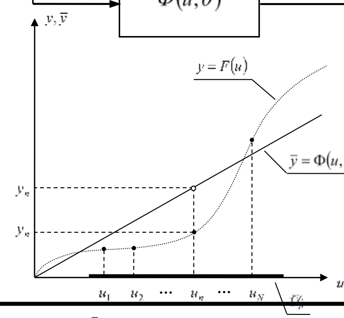


$$y_n = F(u_n, \theta),$$
$$n = 1, 2, \ldots, N$$
$$Y_N = \overline{F}(U_N, \theta)$$
$$\theta = \overline{F}^{-1}(U_N, Y_N) \overset{df}{=} \Psi_N(U_N, Y_N)$$

Choice of the best model



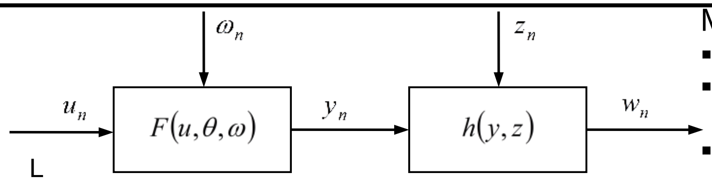
$$\bar{y} = \Phi(u, \theta_N^*)$$
$$\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$$
$$\theta_N^* = \Psi_N(U_N, Y_N)$$
$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$



Optimal model for:

- Given measurements sequences
- Klass of models
- Given performance index

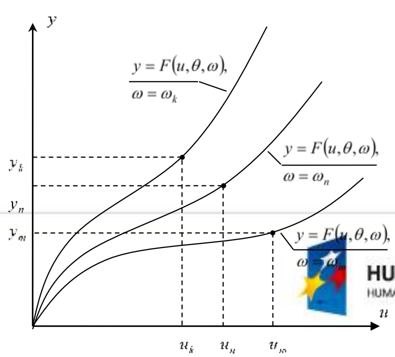
L o s o w y

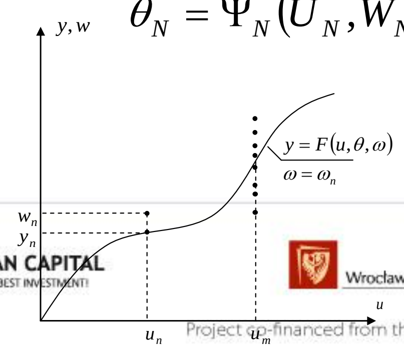


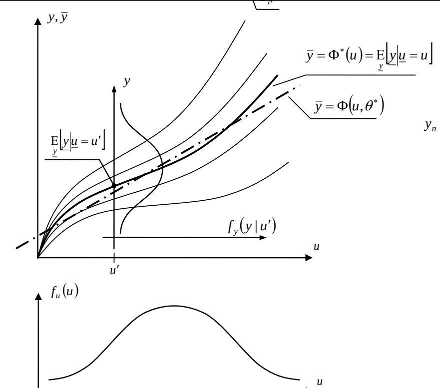
Methods:

- Least square
- Maximum likelihood
- Bayes'a

$$\theta_N = \Psi_N(U_N, W_N)$$

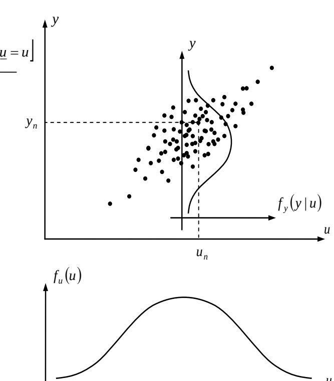






Full probabilistic knowledge

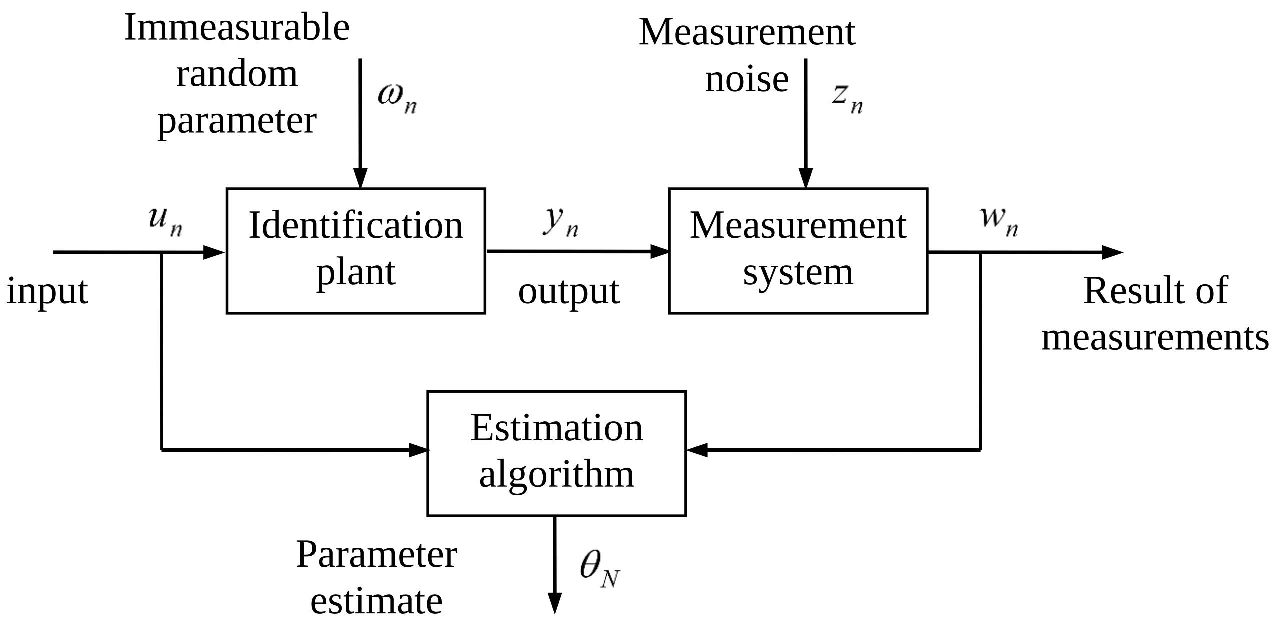
- First type regrsesion
- Second type regression



Unknown probabilostic knowledge

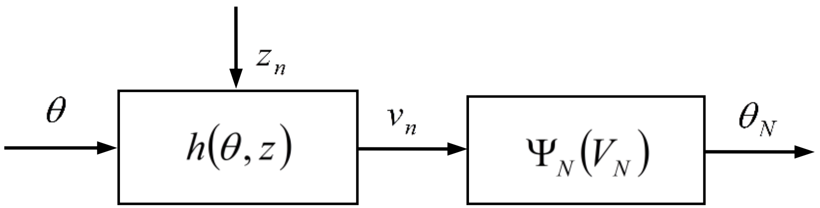
- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

Plant parameter estimation problem



Noised measurements of the physical values

- Problem formulation



<u>Measurement system description:</u> $v = h(\theta, z)$

where: $v \in \mathscr{V}$, h – known one-to-one function

$h : \Theta \times \mathscr{Z} \to \mathscr{V}$, $z = h_z^{-1}(\theta, v)$

examples of h : $v = h(\theta, z) = \theta + z$

$v = h(\theta, z) = \theta \cdot z$

$\mathscr{W}$ – measurements domain ($\dim \theta = \dim z = L$)

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Noised measurements of the physical values

General form of estimation algorithm:

$$\theta_N = \Psi_N(V_N)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayesian method

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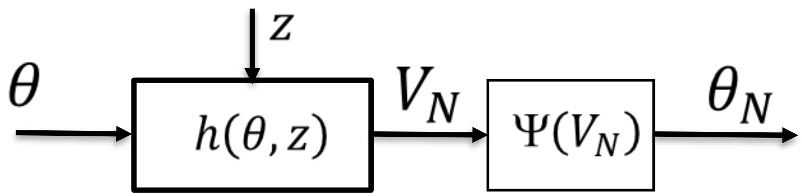
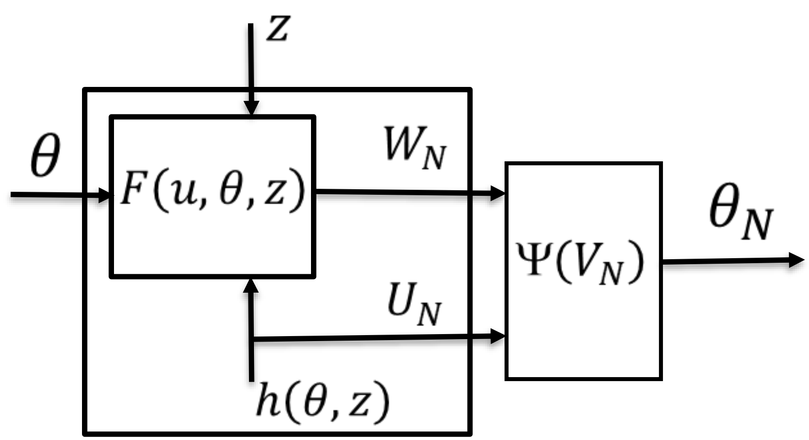
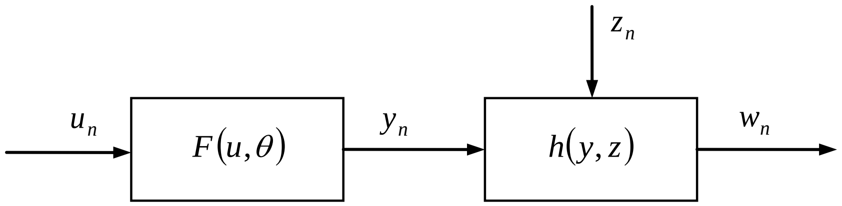


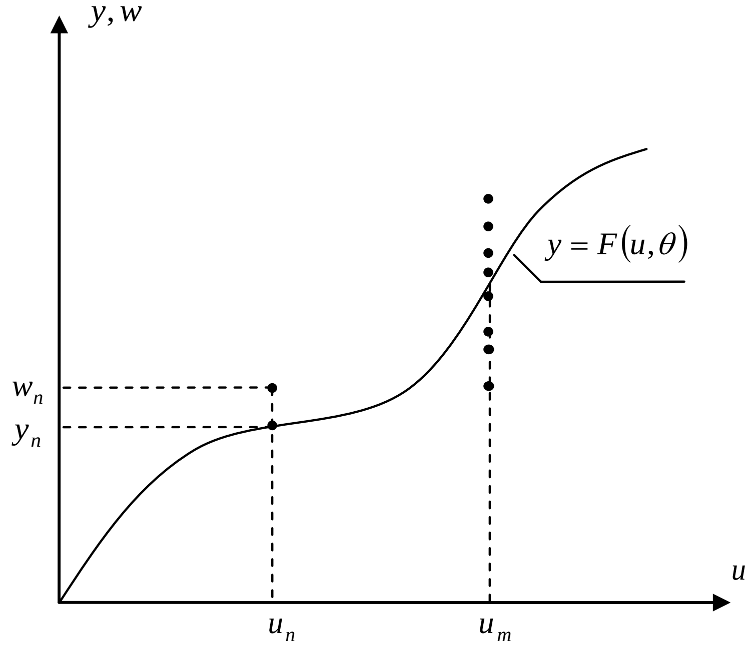


$$V_N \begin{bmatrix} W_N \\ U_N \end{bmatrix}$$

Deterministic plant, noised measurements of the plant output

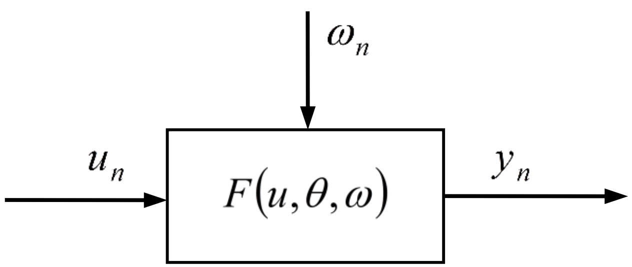
- Noised measurements of the identification plant known static characteristics

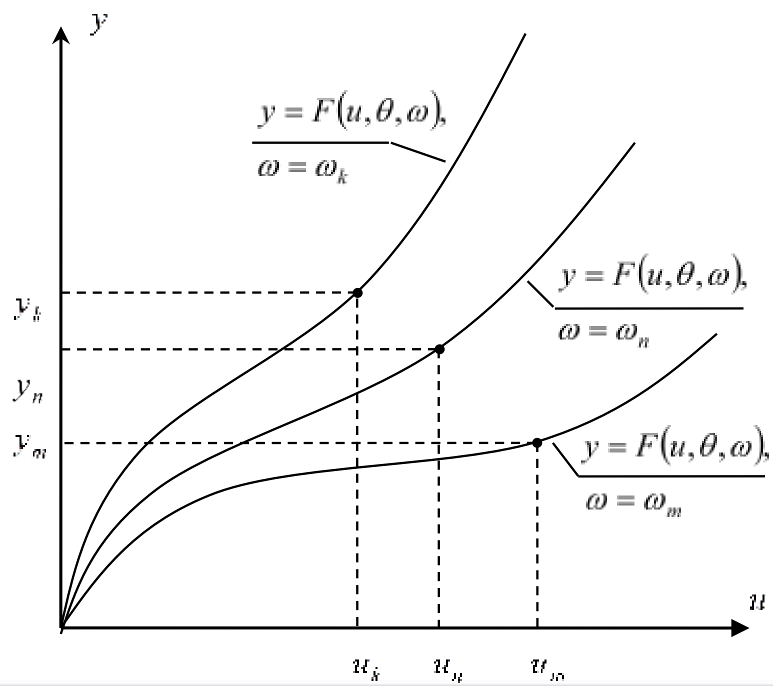




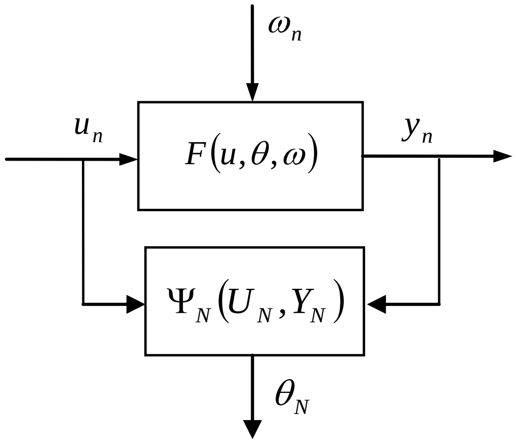
Immeasurable random plant parameter

- Measurements of plant characteristic with random parameter





Immeasurable random plant parameter



- Problem formulation

Plant characteristic: $y = F(u, \theta, \omega)$

Random plant parameter: $\omega \in \Omega \subseteq \mathscr{R}^L$, $(\dim y = \dim \omega = L)$

F – one-to-one mapping $\omega = F_\omega^{-1}(u, \theta, y)$

ω_n – value of random variable $\underline{\omega}$ from $\Omega \subseteq \mathscr{R}^L$

Probability density function $f_\omega(\omega)$ is given

Measurements: $U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$, $Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$

Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N)$

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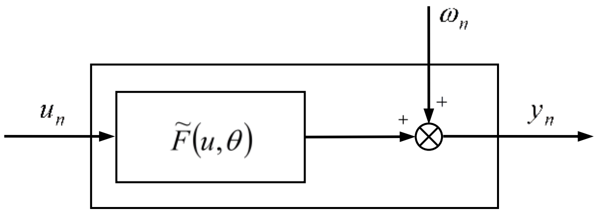
Noised measurements of the physical values

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayesian method

Least square method



Assumptions:

$$y = F(u,\theta,\omega) = \tilde{F}(u,\theta) + \omega$$ – plant characteristic

$$\underset{\underline{\omega}}{E}[\underline{\omega}] = 0 \qquad \underset{\underline{\omega}}{Var}[\underline{\omega}] < \infty$$

Calculations:

Least square method minimizes empirical variance:

$$Var_{\omega N}(U_N, Y_N, \theta) = \frac{1}{N}\sum_{n=1}^{N}\left(y_n - \tilde{F}(u_n,\theta)\right)^2$$

Estimation algorithm has the form:

$$\theta_N = \Psi_N(U_N, Y_N) \quad \rightarrow \quad Var_{\omega N}(U_N, Y_N, \theta_N) = \min_{\theta\in\Theta} Var_{\omega N}(U_N, Y_N, \theta)$$

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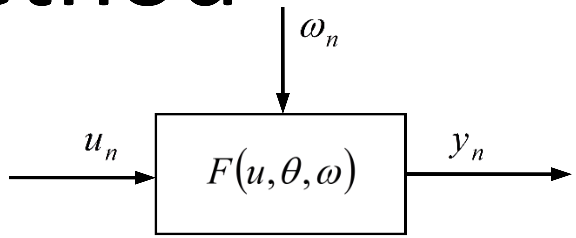
Maximum likelihood method



<u>Assumptions:</u>

ω_n is value of random variable $\underline{\omega}$, with probability density function $f_\omega(\omega)$

For a given input u_n, output $y_n,\ n=1,2,\ldots,N$ is measured

Sequence $y_n,\ n=1,2,\ldots,N$ contains values of random variable $\underline{y}$: $\quad \underline{y}=F(u,\theta,\underline{\omega})$

<u>Calculations:</u>

Probability density function $f_y(y,\theta;u)= f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)\cdot\left|J_F\right|$

where J_F is Jacobi matrix: $J_F=\dfrac{\partial F_\omega^{-1}(u,\theta,y)}{\partial y}$

Likelihood function: $L_N(Y_N,\theta;U_N)=\prod_{n=1}^{N} f_y(y_n,\theta;u_n)=\prod_{n=1}^{N} f_\omega\left(F_\omega^{-1}(u_n,\theta,y_n)\right)\left|J_F\right|$

Estimation algorithm: $\theta_N=\Psi_N(U_N,Y_N) \quad \rightarrow \quad L_N(Y_N,\theta_N;U_N)=\max_{\theta\in\Theta} L_N(Y_N,\theta;U_N)$

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Bayesian method's

Assumptions:

Additionally θ is value of random variable $\underline{\theta}$, with probability density function $f_{\theta}(\theta)$

Calculations:

$$r(\bar{\theta},Y_N;U_N) \stackrel{df}{=} \underset{\underline{\theta}}{E}\left[L(\underline{\theta},\bar{\theta})\middle|Y_N;U_N\right] = \int_{\Theta} L(\theta,\bar{\theta}=\bar{\Psi}(U_N,Y_N))\, f'(\theta|Y_N;U_N)\,d\theta$$

$$f_y(y|\theta;u) = f_{\omega}\left(F_{\omega}^{-1}(u,\theta,y)\right)|J_F|$$

A'posteriori probability density function:

$$f'(\theta|Y_N;U_N) = \frac{f_{\theta}(\theta)\cdot\prod_{n=1}^{N} f_{\omega}\left(F_{\omega}^{-1}(u_n,\theta,y_n)\right)|J_F|}{\int_{\Theta} f_{\theta}(\theta)\cdot\prod_{n=1}^{N} f_{\omega}\left(F_{\omega}^{-1}(u_n,\theta,y_n)\right)|J_F|\,d\theta}$$

Estimation algorithm: $\theta_N = \Psi_N(U_N,Y_N) \quad \rightarrow \quad r(\theta_N,Y_N;U_N) = \min_{\bar{\theta}\in\Theta} r(\bar{\theta},Y_N;U_N)$

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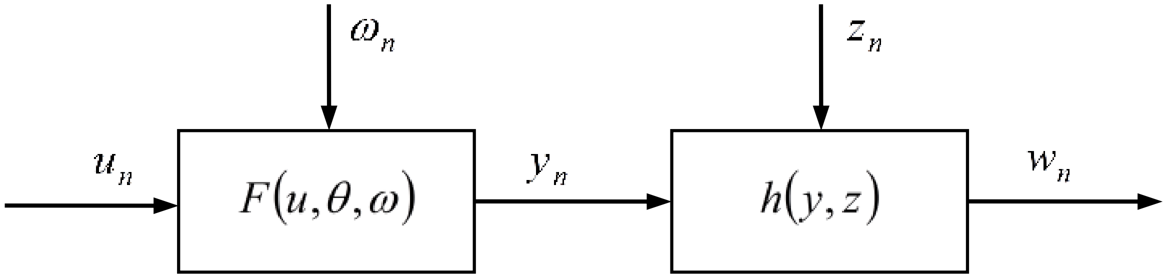
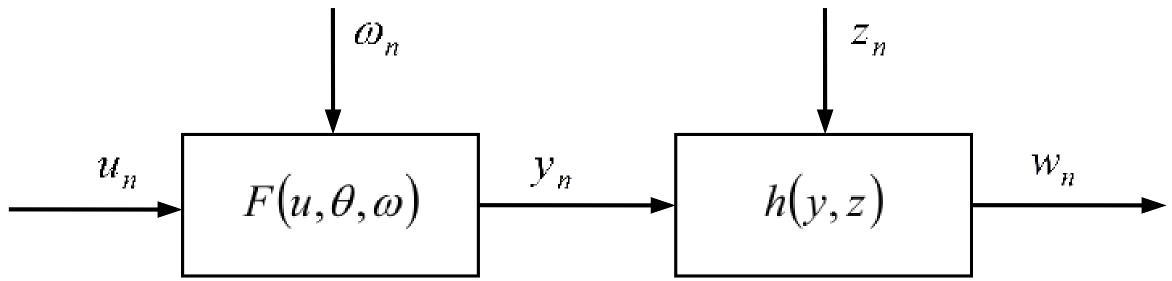
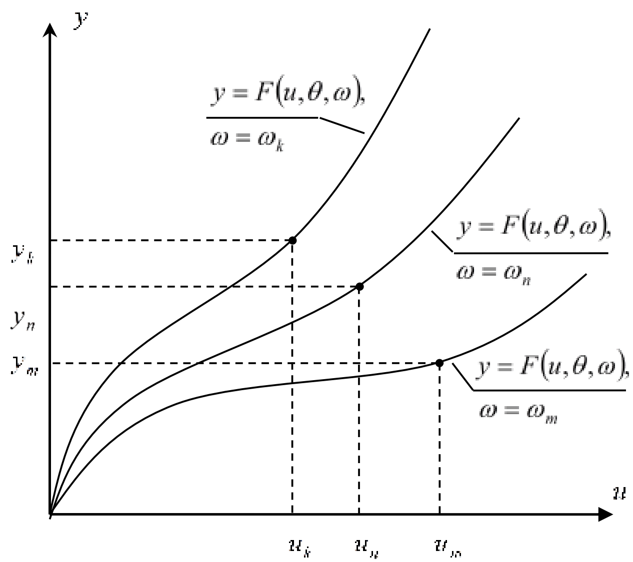
Random plant parameter and measurement noise

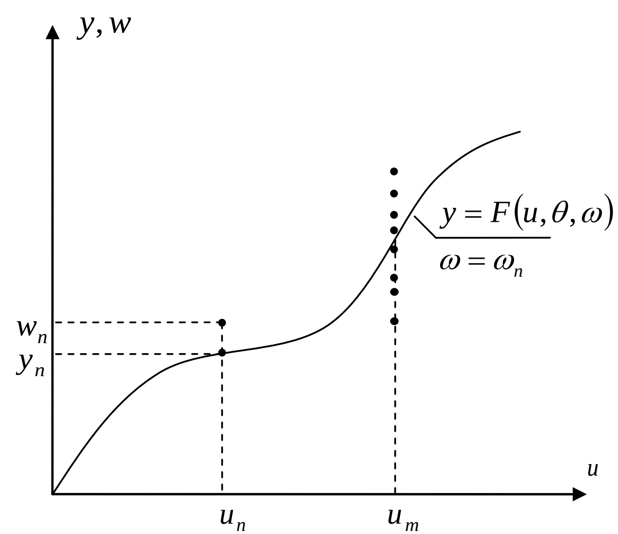
- Noised measurement plant output with randomly changed parameters



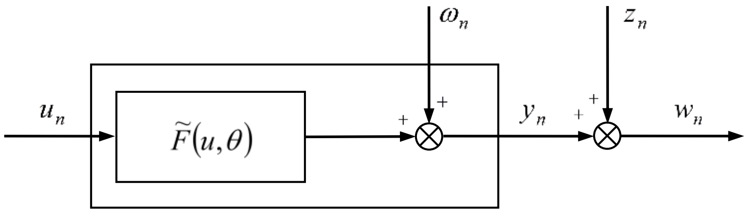
- Noised measurement plant output with randomly changed parameters







Least square method



Assumptions:

$\underline{z}, \ \underline{\omega}$ – random variables such that $\underset{\underline{\omega}}{E}[\underline{\omega}]=0, \ \underset{\underline{\omega}}{Var}[\underline{\omega}]<\infty, \ \underset{\underline{z}}{E}[\underline{z}]=0, \ \underset{\underline{z}}{Var}[\underline{z}]<\infty$

Measurements are values of random variable: $\underline{w}=\widetilde{F}(u,\theta)+\underline{\omega}+\underline{z}$

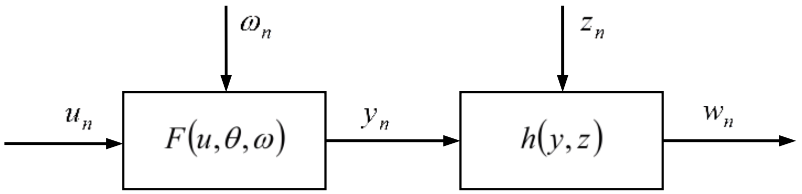
$$\underset{\underline{\omega}+\underline{z}}{E}[\underline{\omega}+\underline{z}]=0, \ \underset{\underline{\omega}+\underline{z}}{Var}[\underline{\omega}+\underline{z}]<\infty$$

Calculations:

Empirical variance: $Var_{(\omega+z)N}(U_N,W_N,\theta)=\frac{1}{N}\sum_{n=1}^{N}\left(w_n-\widetilde{F}(u_n,\theta)\right)^2$

Estimation algorithm: $\theta_N=\Psi_N(U_N,Y_N) \ \rightarrow \ Var_{(\omega+z)N}(U_N,W_N,\theta_N)=\min_{\theta\in\Theta} Var_{(\omega+z)N}(U_N,W_N,\theta)$

Maximum likelihood method



<u>Assumptions:</u>

ω_n is value of random variable $\underline{\omega}$, with probability density function $f_\omega(\omega)$

z_n is value of random variable $\underline{z}$, with probability density function $f_z(z)$

$w_n,\ n=1,2,\ldots,N$ are values of random variable $\underline{w}=h(\underline{y},\underline{z})$

For a given $u_n,\ y_n,\ n=1,2,\ldots,N$ are values of random variable $\underline{y}=F(u,\theta,\underline{\omega})$

<u>Calculations:</u>

Function h is no one to one mapping with respect $(\underline{y},\underline{z})$, so we add identity $\underline{y}=\underline{y}$ and we have:

$$\underline{w}=h(\underline{y},\underline{z})$$
$$\underline{y}=\underline{y}$$

The inverse transformation:

$$\underline{z}=h_z^{-1}(\underline{y},\underline{w})$$
$$\underline{y}=\underline{y}$$

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Maximum likelihood method

Jacobi matrix: $$J = \begin{bmatrix} \dfrac{\partial h_z^{-1}(y,w)}{\partial w} & \dfrac{\partial h_z^{-1}(y,w)}{\partial y} \\ \dfrac{\partial y}{\partial w} & \dfrac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial h_z^{-1}(y,w)}{\partial w} & \dfrac{\partial h_z^{-1}(y,w)}{\partial y} \\ O_{L\times L} & I_{L\times L} \end{bmatrix}$$

where: $O_{L\times L}$ is *L* – dimensional zero matrix, $I_{L\times L}$ is *L* – dimensional unit matrix

Determinant of Jacobi matrix:

$$|J| = \left\| \begin{bmatrix} \dfrac{\partial h_z^{-1}(y,w)}{\partial w} & \dfrac{\partial h_z^{-1}(y,w)}{\partial y} \\ O_{L\times L} & I_{L\times L} \end{bmatrix} \right\| = \left| \dfrac{\partial h_z^{-1}(y,w)}{\partial w} \right| = |J_h|$$

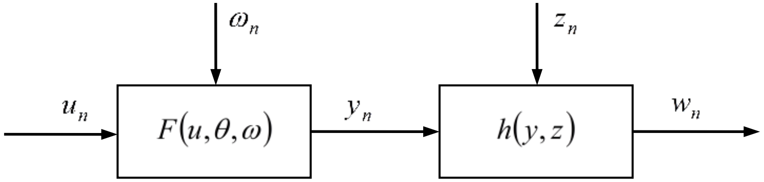
Maximum likelihood method

Joint probability density function of random variables ($\underline{w}$,$\underline{y}$):

$$f_w(w,y,\theta;u) = f_z\left(h_z^{-1}(y,w)\right)f_y(y,\theta;u)\left|J\right| = f_z\left(h_z^{-1}(y,w)\right)f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)\left|J_F\right|\left|J_h\right|$$

Marginal probability density function of random wariable $\underline{w}$:

$$f_w(w,\theta;u) = \int\limits_{\mathcal{Y}} f_z\left(h_z^{-1}(y,w)\right)f_y(y,\theta;u)\left|J\right|dy = \int\limits_{\mathcal{Y}} f_z\left(h_z^{-1}(y,w)\right)f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)\left|J_F\right|\left|J_h\right|dy$$



Likelihood function:

$$L_N(W_N,\theta;U_N) = \prod_{n=1}^{N} f_w(w_n,\theta;u_n) =$$

$$= \prod_{n=1}^{N} \int\limits_{\mathcal{Y}} f_z\left(h_z^{-1}(y,w_n)\right)f_\omega\left(F_\omega^{-1}(u_n,\theta,y)\right)\left|J_F\right|\left|J_h\right|dy$$

Estimation algorithm: $\theta_N = \Psi_N(U_N,Y_N) \;\; \rightarrow \;\; L_N(W_N,\theta_N;U_N) = \max\limits_{\theta\in\Theta} L_N(W_N,\theta;U_N)$

Bayesian method’s

Assumptions:

Additionally θ is value of random variable $\underline{\theta}$, with probability density function $f_\theta(\theta)$

For a given parametr θ and input u_n the sequence $y_n, n=1,2,\ldots,N$ are values of random variable $\underline{y}$ under condition $\underline{\theta}=\theta$ and input is equal u.

$w_n, n=1,2,\ldots,N$ are values of random variable $\underline{w}$ under condition $\underline{\theta}=\theta$

z_n is value of random variable $\underline{z}$, with probability density function $f_z(z)$

Calculations:

$$\underline{w}=h(\underline{y},\underline{z})$$

$$f_w(w|\theta;u)=\int_{\mathscr{Y}} f_z\left(h_z^{-1}(y,w)\right)f_y(y|\theta;u)|J|dy=\int_{\mathscr{Y}} f_z\left(h_z^{-1}(y,w)\right)f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)|J_F||J_h|dy$$

Bayesian method's

A'posteriori probability density function:

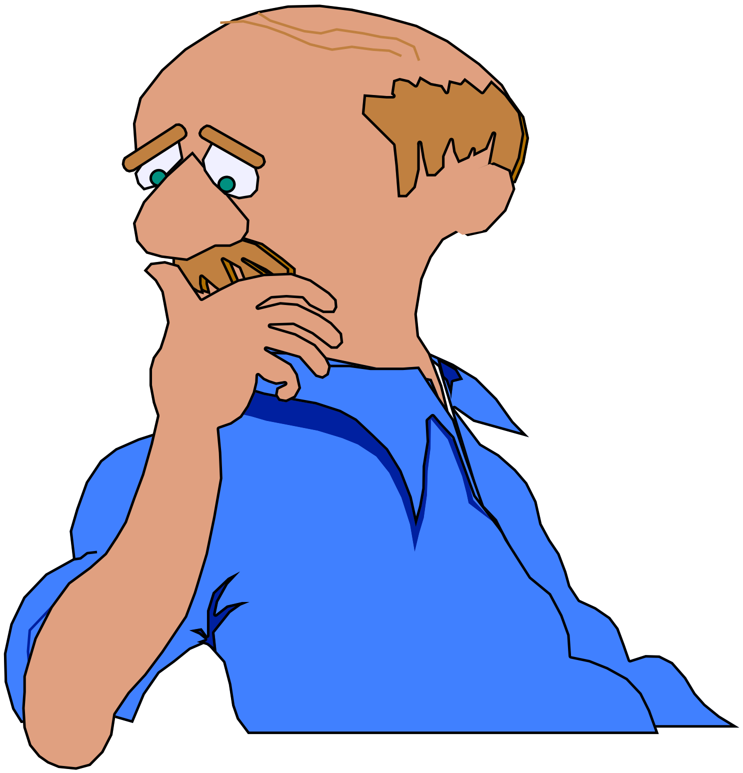
$$f'(\theta|W_N;U_N)=\frac{f_\theta(\theta)\prod_{n=1}^{N}\int_{\mathcal{Y}} f_z\left(h_z^{-1}(y,w_n)\right)f_\omega\left(F_\omega^{-1}(u_n,\theta,y)\right)|J_F||J_h|dy}{\int_\Theta f_\theta(\theta)\prod_{n=1}^{N}\int_{\mathcal{Y}} f_z\left(h_z^{-1}(y,w_n)\right)f_\omega\left(F_\omega^{-1}(u_n,\theta,y)\right)|J_F||J_h|dy\,d\theta}$$

Conditional risk :

$$r(\bar{\theta},Y_N;U_N)\overset{\mathrm{df}}{=}\underset{\underline{\theta}}{E}\left[L(\underline{\theta},\bar{\theta})|W_N;U_N\right]=\int_\Theta L\left(\theta,\bar{\theta}=\overline{\Psi}(U_N,W_N)\right)f'(\theta|W_N;U_N)\,d\theta$$

Estimation algorithm: $\theta_N=\Psi_N(U_N,W_N)\quad\rightarrow\quad r(\theta_N,W_N;U_N)=\min_{\bar{\theta}\in\Theta} r(\bar{\theta},W_N;U_N)$

Thank you for attention



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Least square method

$$w_n = u_n + z_n, \quad u_n = R i_n, \; n = 1,2,\cdots,N$$

$$\bar{z} \sim \mathrm{N}(0, \sigma_z)$$

$$Var_N(R) = \sum_{n=1}^{N} (w_n - R i_n)^2$$

$$R_N = \frac{\sum_{n=1}^{N} w_n i_n}{\sum_{n=1}^{N} (i_n)^2}$$

Maximum likelihood method

$w_n = u_n + z_n, \quad u_n = Ri_n \quad \bar{z} \sim \mathrm{N}(m_z, \sigma_z)$

$$f_z(z) = \frac{1}{\sqrt{2\pi}\sigma_z} exp\left(-\frac{(z-m_z)^2}{2\sigma_z^2}\right) \qquad f_R(R) = \frac{1}{\sqrt{2\pi}\sigma_R} exp\left(-\frac{(R-m_R)^2}{2\sigma_R^2}\right)$$

$z_n = w_n - Ri_n, \quad J = 1$

$$f_w(w, R; i) = \frac{1}{\sqrt{2\pi}\sigma_z} exp\left(-\frac{(w - Ri - m_z)^2}{2\sigma_z^2}\right)$$

$$L_N(W_N, R; I_N) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_z} exp\left(-\frac{(w_n - Ri_n - m_z)^2}{2\sigma_z^2}\right) =$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma_z}\right)^N exp\left(-\frac{\sum_{n=1}^{N}(w_n - Ri_n - m_z)^2}{2\sigma_z^2}\right)$$

$$R_N = \frac{\sum_{n=1}^{N}(w_n - m_z)i_n}{\sum_{n=1}^{N}(i_n)^2}$$

$$R_N = \frac{\sum_{n=1}^{N}(w_n - m_z)i_n}{\sum_{n=1}^{N}(i_n)^2}$$

Bayesian method's

$w_n = u_n + z_n, \quad u_n = Ri_n \quad \bar{z} \sim \mathrm{N}(m_z, \sigma_z) \; \bar{R} \sim \mathrm{N}(m_R, \sigma_R) \quad L(R, \bar{R}) = -\delta(R - \bar{R})$

$$f_z(z) = \frac{1}{\sqrt{2\pi}\sigma_z} exp\left(-\frac{(z - m_z)^2}{2\sigma_z^2}\right) \qquad f_R(R) = \frac{1}{\sqrt{2\pi}\sigma_R} exp\left(-\frac{(R - m_R)^2}{2\sigma_R^2}\right)$$

$z_n = w_n - Ri_n, \qquad J = 1$

$$f_w(w; i) = \frac{1}{\sqrt{2\pi}\sigma_z} exp\left(-\frac{(w - Ri - m_z)^2}{2\sigma_z^2}\right)$$

$$r(R, W_N; I_N) = -\frac{1}{\sqrt{2\pi}\sigma_R} exp\left(-\frac{(R-m_R)^2}{2\sigma_R^2}\right) \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_z} exp\left(-\frac{(w_n - Ri_n - m_z)^2}{2\sigma_z^2}\right) =$$

$$= -\frac{1}{\sqrt{2\pi}\sigma_R}\left(\frac{1}{\sqrt{2\pi}\sigma_z}\right)^N exp\left(-\frac{(R-m_R)^2}{2\sigma_R^2} - \frac{\sum_{n=1}^{N}(w_n - Ri_n - m_z)^2}{2\sigma_z^2}\right)$$

$$R_N = \frac{m_R + \left(\frac{\sigma_R}{\sigma_z}\right)^2 \sum_{n=1}^{N} w_n i_n}{1 + \left(\frac{\sigma_R}{\sigma_z}\right)^2 \sum_{n=1}^{N} (i_n)^2}$$

Immeasurable random plant parameter

$$u_n = Ri_n, \;\; n = 1,2,\cdots,N$$

$$R = R_0 + \beta T = R_0 + \omega, \qquad \omega = \beta T \quad \omega \sim N(m_\omega, \sigma_\omega)$$

$$f_\omega(\omega) = \frac{1}{\sqrt{2\pi}\sigma_\omega} exp\left(-\frac{(\omega - m_\omega)^2}{2\sigma_\omega^2}\right)$$

$$u = Ri = (R_0 + \omega)i \;\; \rightarrow \;\; \omega = \frac{u}{i} - R_0, \;\; J = \frac{1}{i}$$

$$R_{0N} = \frac{1}{N}\sum_{n=1}^{N}\frac{u_n}{i_n} - m_\omega$$

$$f_u(u; i.R_0) = \frac{1}{\sqrt{2\pi}\sigma_\omega} exp\left(-\frac{\left(\frac{u}{i} - R_0 - m_\omega\right)^2}{2\sigma_\omega^2}\right)\frac{1}{i}$$

$$L_N(U_N, R_0; I_n) = \prod_{n=1}^{N} f_u(u_n; i_n.R) = \prod_{n=1}^{N}\frac{1}{\sqrt{2\pi}\sigma_\omega} exp\left(-\frac{\left(\frac{u_n}{i_n} - R_0 - m_\omega\right)^2}{2\sigma_\omega^2}\right)\frac{1}{i_n} =$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma_\omega}\right)^N exp\left(-\frac{\sum_{n=1}^{N}\left(\frac{u_n}{i_n} - R_0 - m_\omega\right)^2}{2\sigma_\omega^2}\right)\frac{1}{\prod_{n=1}^{N} i_n}$$

$$L_N(U_N, R_0; I_n) = \prod_{n=1}^{N} f_u(u_n; i_n . R) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_\omega} exp\left(-\frac{\left(\frac{u_n}{i_n} - R_0 - m_\omega\right)^2}{2\sigma_\omega^2}\right)\frac{1}{i_n} =$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma_\omega}\right)^N exp\left(-\frac{\sum_{n=1}^{N}\left(\frac{u_n}{i_n} - R_0 - m_\omega\right)^2}{2\sigma_\omega^2}\right)\frac{1}{\prod_{n=1}^{N} i_n}$$

$$R_{0N} = \frac{\sum_{n=1}^{N}\left(\frac{u_n}{i_n} - m_\omega\right)}{N}$$

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Thank you for attention