Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.3d. Choice of the best model, probabilistic case

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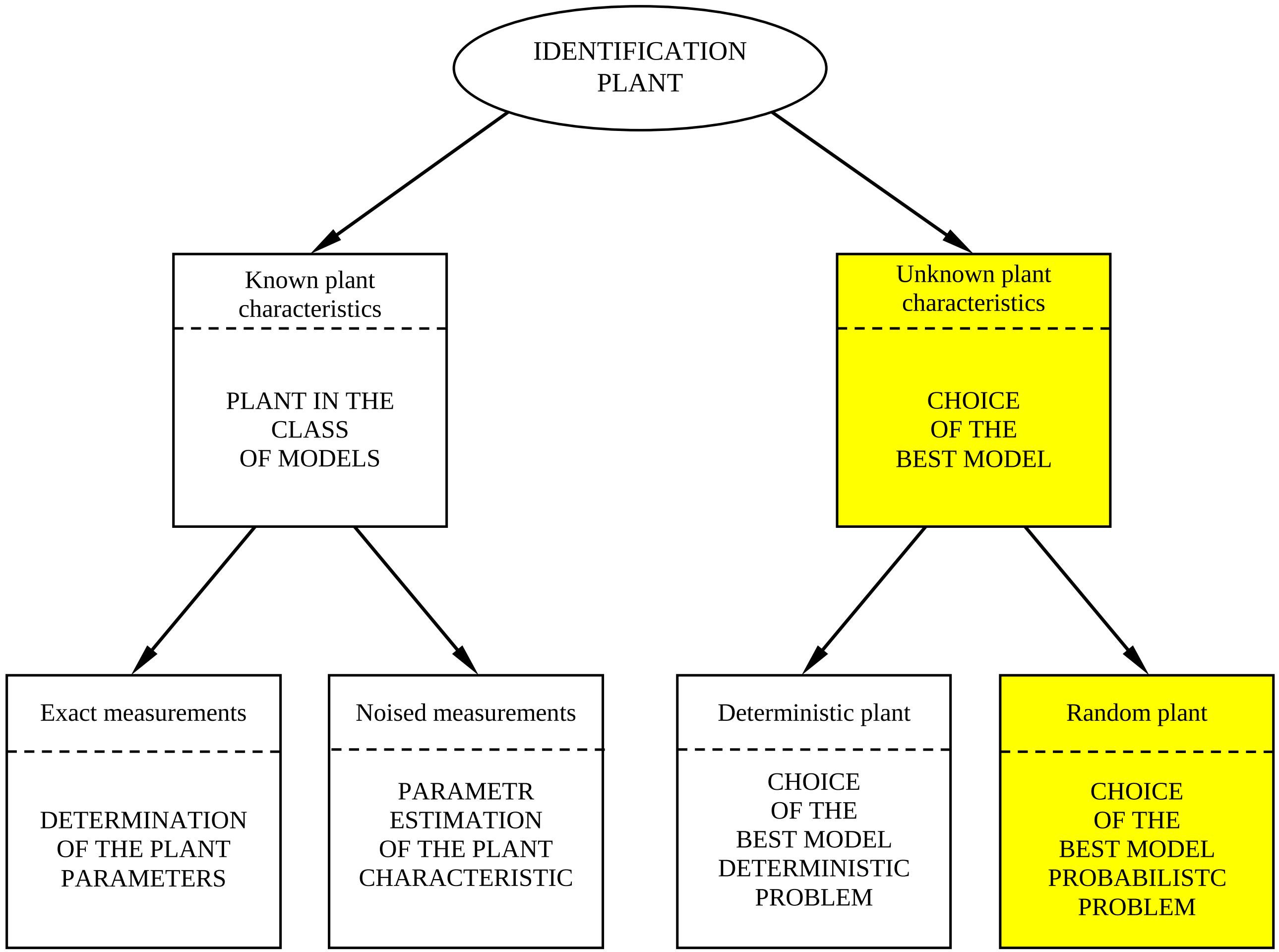
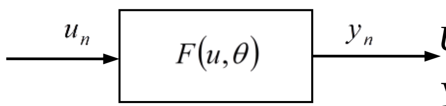
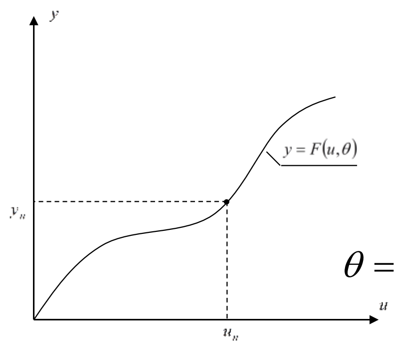

IDENTIFICATION
PLANT
Known plant characteristics
PLANT IN THE CLASS OF MODELS
Unknown plant characteristics
CHOICE OF THE BEST MODEL
Exact measurements
DETERMINATION OF THE PLANT PARAMETERS
Noised measurements
PARAMETR ESTIMATION OF THE PLANT CHARACTERISTIC
Deterministic plant
CHOICE OF THE BEST MODEL DETERMINISTIC PROBLEM
Random plant
CHOICE OF THE BEST MODEL PROBABILISTC PROBLEM

Plant in the class of model

D e t e r m i n i s t y c z n y

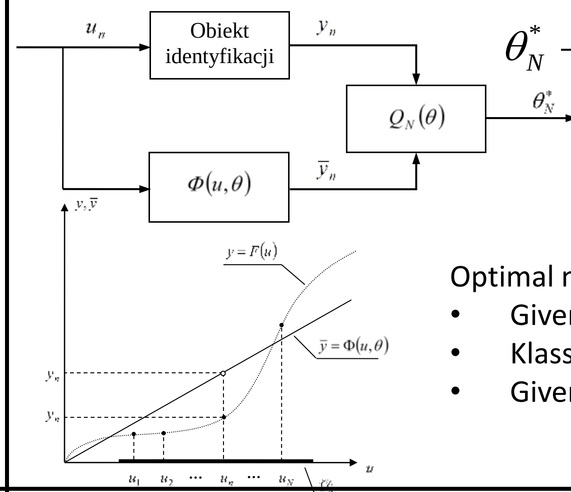


$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$



$$y_n = F(u_n, \theta),$$
$$n = 1, 2, \ldots, N$$
$$Y_N = \overline{F}(U_N, \theta)$$
$$\theta = \overline{F}^{-1}(U_N, Y_N) \overset{df}{=} \Psi_N(U_N, Y_N)$$

Choice of the best model

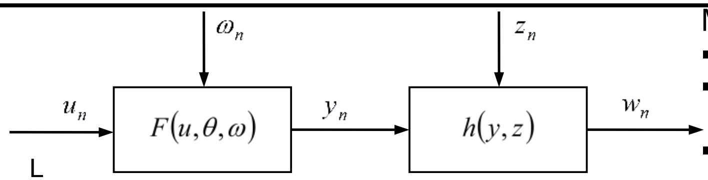


$$\bar{y} = \Phi(u, \theta_N^*)$$
$$\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$$
$$\theta_N^* = \Psi_N(U_N, Y_N)$$
$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

Optimal model for:

- Given measurements sequences
- Klass of models
- Given performance index

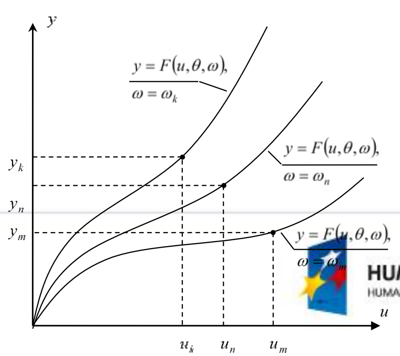
L o s o w y

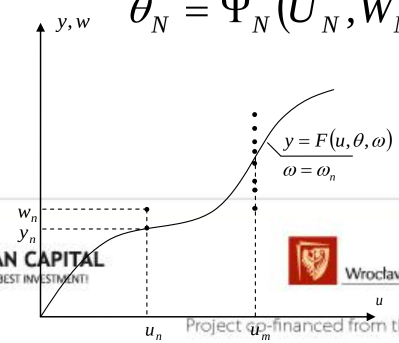


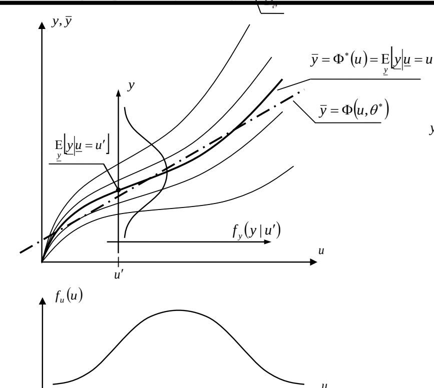
Methods:

- Least square
- Maximum likelihood
- Bayes'a

$$\theta_N = \Psi_N(U_N, W_N)$$

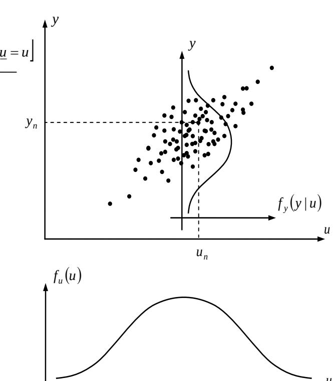






Full probabilistic knowledge

- First type regrsesion
- Second type regression



Unknown probabilostic knowledge

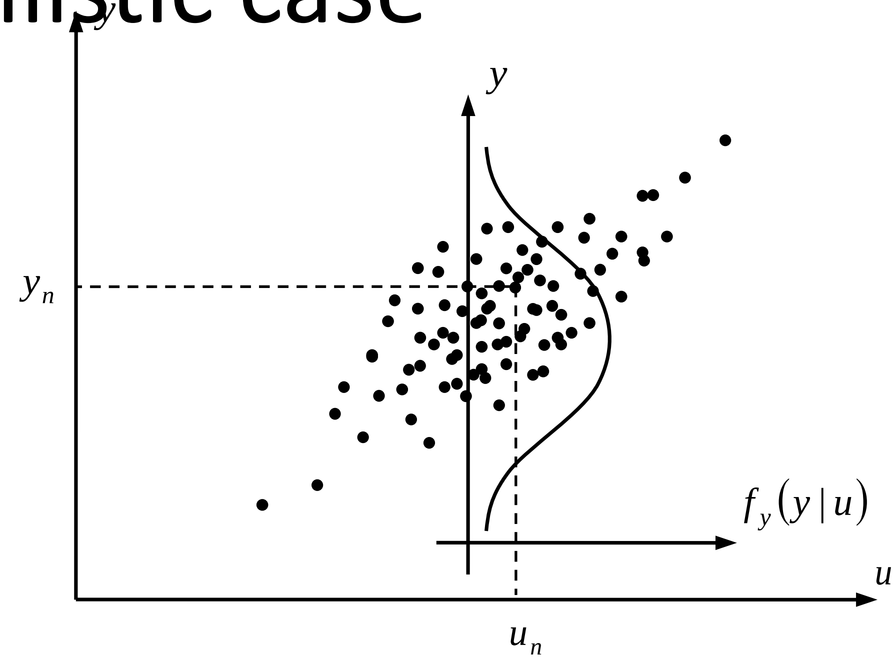
- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

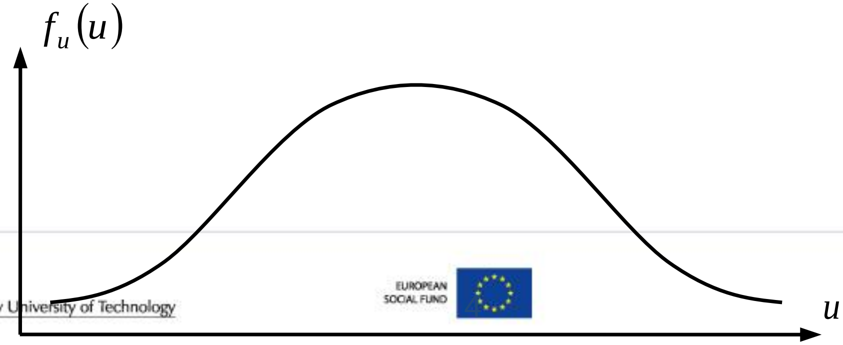
Choice of the best model, probabilistic case

$$(u_n, y_n),\ n = 1, 2, \ldots, N$$

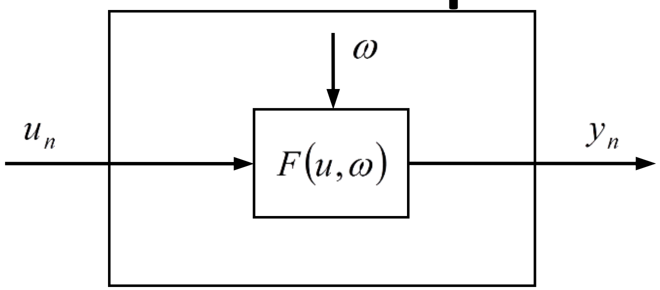
are values of random variables

$$(\underline{u}, \underline{y})$$



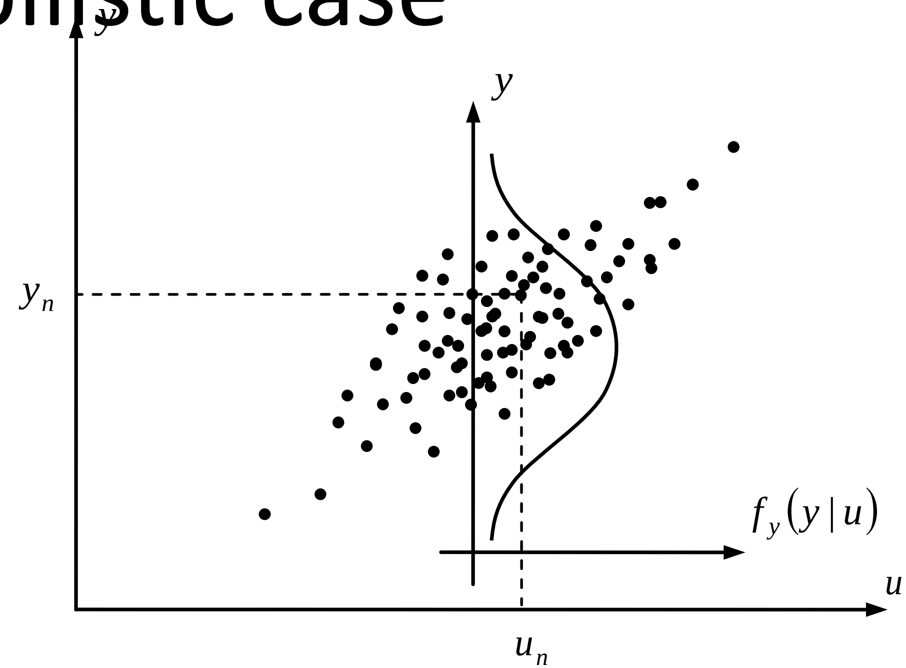


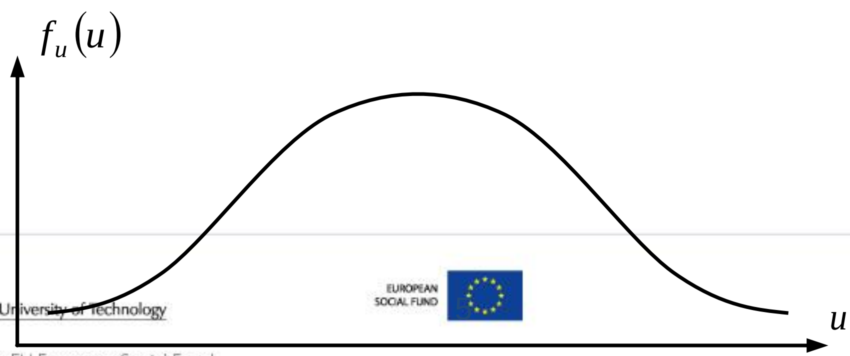
Choice of the best model, probabilistic case



$$(u_n, y_n),\ n = 1, 2, \ldots, N$$

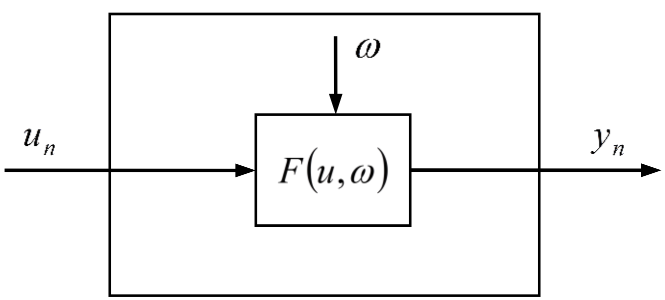
are values of random variables $(\underline{u}, \underline{y})$

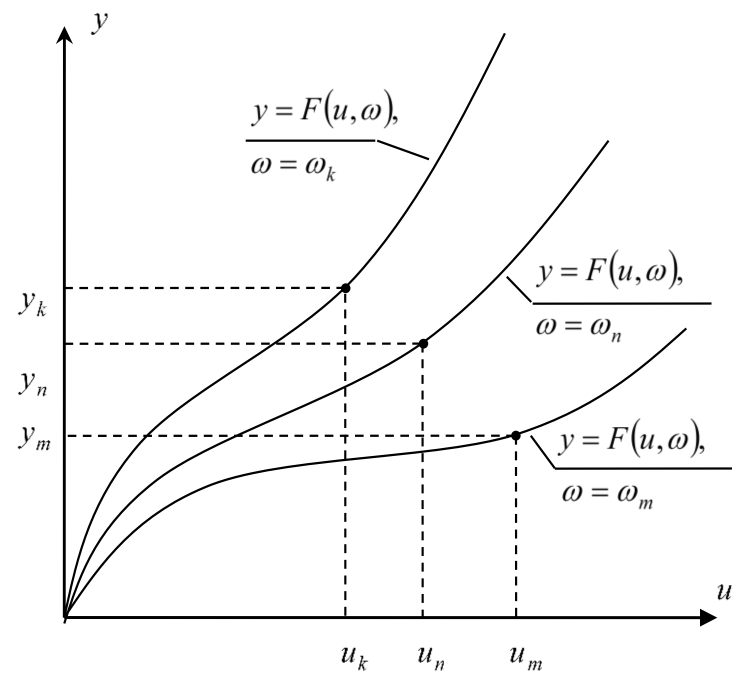




Choice of the best model, probabilistic case

- Plant identification with random immeasurable variable





Choice of the best model, probabilistic case

- Problem formulation

Plant characteristic: $\underline{y} = F(u, \underline{\omega})$

ω_n – value of random variable – plant parameter $\underline{\omega}$ $\left(\omega \in \Omega \subseteq \mathscr{R}^L\right)$, $f_\omega(\omega)$

y_n – value of random variable $\underline{y}$ – transformation of variable $\underline{\omega}$

F – invertible function with respect ω : $\underline{\omega} = F_\omega^{-1}(u, \underline{y})$

Choice of the best model, probabilistic case

Probability density function $\underline{y}$ under condition $\underline{u} = u$:

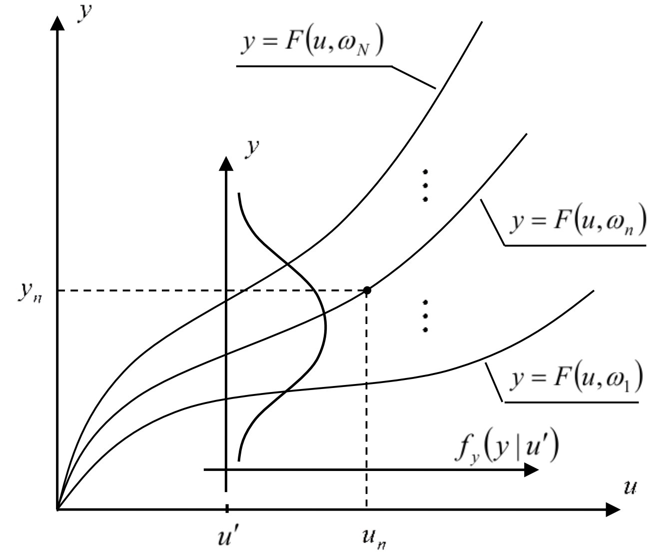
$$f_y(y \mid u) = f_\omega\left(F_\omega^{-1}(u, y)\right) \cdot \left|J_F\right|$$

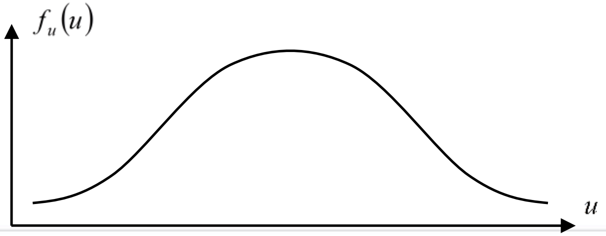
where
$$J_F = \frac{\partial F_\omega^{-1}(u, y)}{\partial \underline{y}}$$

u_n – value of random variable $\underline{u}$, $f_u(u)$

input u_n and output y_n are values of the random variables $(\underline{u}, \underline{y})$

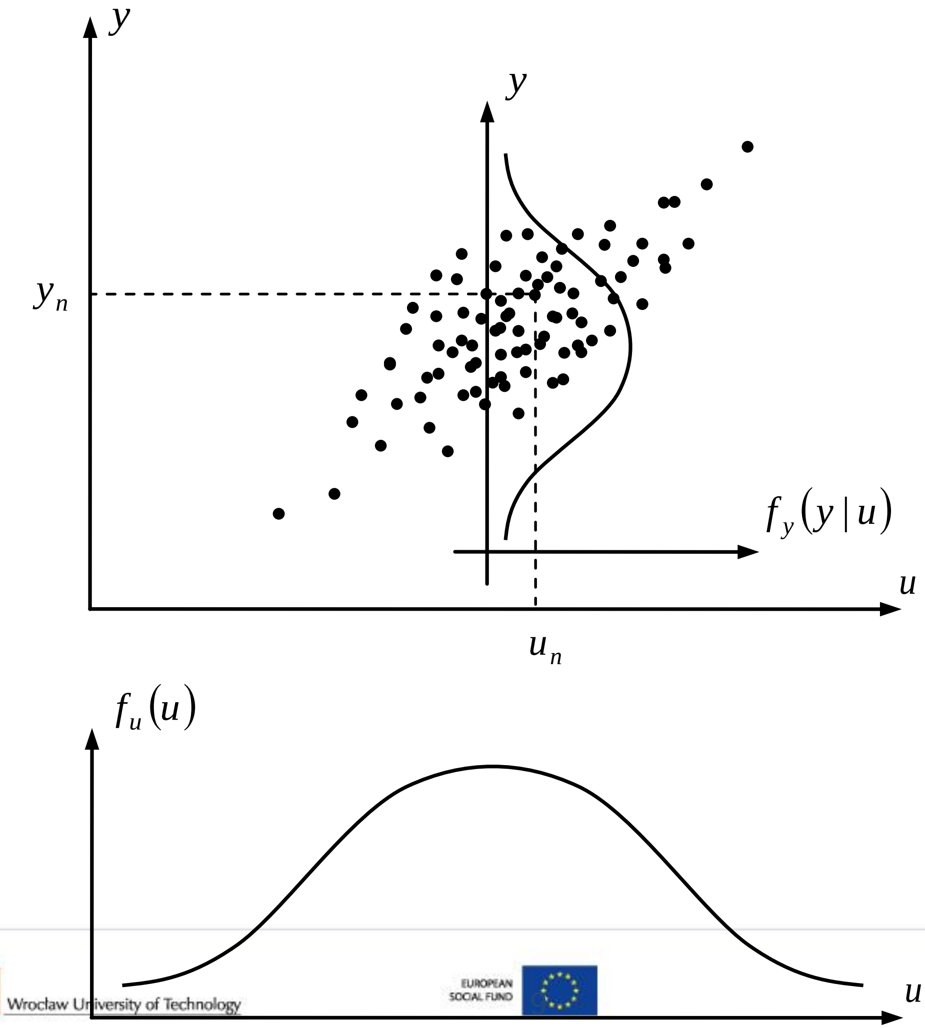
$$u \in \mathscr{U} \subseteq \mathscr{R}^S, \quad y \in \mathscr{Y} \subseteq \mathscr{R}^L$$





$$(u_n, y_n),\ n = 1, 2, \ldots, N$$

are values of random variables $(\underline{u}, \underline{y})$



Choice of the best model, probabilistic case

Two possible cases

Full a'priori knowledge

- joint probability density function $f(u, y)$ of random variables $(\underline{u}, \underline{y})$

or

- conditional probability density function $f_y(y|u)$

and mrginal probability density function

$$f_u(u)$$

are known

Incomplete probabilistic information

joint probability density function of random variables $(\underline{u}, \underline{y})$ exist, but is not known.
Measurements:

$$(u_n, y_n),\ n = 1, 2, \ldots, N$$

are values of $(\underline{u}, \underline{y})$

Full a'priori knowledge

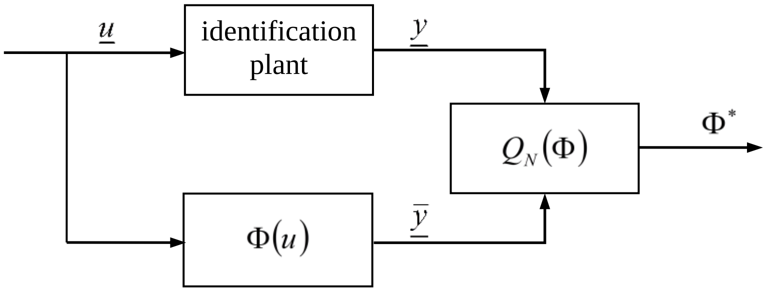
$$f(u,y) = f_y(y|u) \times f_u(u)$$

Model: $\bar{y} = \Phi(u)$

Performance index: $Q(\Phi) = \underset{\underline{u},\underline{y}}{E}\left[q\left(\underline{y}, \Phi(\underline{u})\right)\right] = \int_{\mathcal{U}} \int_{\mathcal{Y}} q(y, \Phi(u)) f(u,y)\, dy\, du$

Optimal model: $\Phi^* \rightarrow \quad Q(\Phi^*) = \min_{\Phi} Q(\Phi)$

$$\bar{y} = \Phi^*(u)$$



Full a'priori knowledge

$$Q(\Phi) = \underset{\underline{u},}{E}\left[\underset{\underline{y}}{E}\left[q\left(\underline{y}, \Phi(\underline{u})\right) \middle| \underline{u} = u\right]\right] = \int\limits_{\mathscr{U}}\left[\int\limits_{\mathscr{Y}} q(y, \Phi(u)) f_y(y|u)\, dy\right] f_u(u)\, du$$

For given input u:

$$Q_u(\Phi(u)) = \underset{\underline{y}}{E}\left[q\left(\underline{y}, \Phi(\underline{u})\right) \middle| \underline{u} = u\right] = \int\limits_{\mathscr{Y}} q(y, \Phi(u)) f_y(y|u)\, dy$$

$$\Phi^*(u) \rightarrow \quad Q_u\left(\Phi^*(u)\right) = \min_{\Phi(u)} Q_u(\Phi(u))$$

For: $q(y, \bar{y}) = [y - \bar{y}]^T [y - \bar{y}]$

Performance index: $Q_u(\Phi(u)) = \underset{\underline{y}}{E}\left[[y - \Phi(u)]^T [y - \Phi(u)] \middle| \underline{u} = u\right] = \int\limits_{\mathscr{Y}} [y - \Phi(u)]^T [y - \Phi(u)] f_y(y|u)\, dy$

$$Q_u(\Phi(u)) = \underset{\underline{y}}{E}\left[[\underline{y} - \Phi(u)]^T[\underline{y} - \Phi(u)] \middle| \underline{u} = u\right] = \int_{\mathcal{Y}} [y - \Phi(u)]^T [y - \Phi(u)] f_y(y|u)\, dy$$

$$Q(\Phi(u)) = E_{\underline{y}}\left[\underline{y}^T \underline{y} | u\right] - 2\Phi(u) E_{\underline{y}}\left[\underline{y} | u\right] + \Phi(u)^T \Phi(u)$$

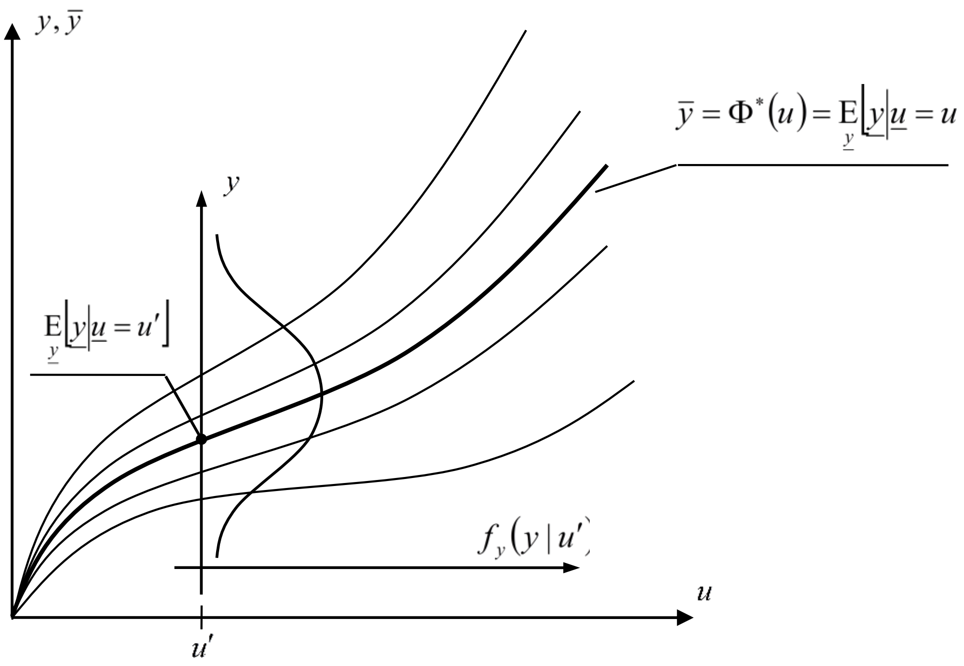
$$\nabla_{\Phi(u)} Q(\Phi(u)) = -2E_{\underline{y}}\left[\underline{y} | u\right] + 2\Phi^*(u) = 0_L$$

$$\Phi^*(u) = E_{\underline{y}}\left[\underline{y} | u\right] = \int_{\mathcal{Y}} y f_y(y|u) dy$$

Full a'priori knowledge

- Regression of the I type

$$\bar{y} = \Phi^*(u) = \underset{\underline{y}}{E}\left[\underline{y} \middle| \underline{u} = u\right] = \int_{\mathscr{Y}} y \; f\left(y \middle| u\right) dy$$



Full a'priori knowledge

- Example

Joint probability density function of random variables $(\underline{u}, \underline{y})$:

$$f(u,y) = (2\pi)^{-\frac{L+S}{2}} \left|\Sigma^{-1}\right|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\begin{bmatrix} u \\ y \end{bmatrix} - \begin{bmatrix} m_u \\ m_y \end{bmatrix}\right)^T \Sigma^{-1}\left(\begin{bmatrix} u \\ y \end{bmatrix} - \begin{bmatrix} m_u \\ m_y \end{bmatrix}\right)\right]$$

Σ – covariance matrix: $\Sigma = \begin{bmatrix} \Sigma_{uu} & \Sigma_{uy} \\ \Sigma_{yu} & \Sigma_{yy} \end{bmatrix}$

Conditional probability density function:

$$f_y(y|u) = (2\pi)^{-\frac{L}{2}} \left|\Sigma_{y|u}^{-1}\right|^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(y - m_{y|u}\right)^T \Sigma_{y|u}^{-1}\left(y - m_{y|u}\right)\right]$$

Full a'priori knowledge

- Example

Covariance matrix $\Sigma_{y|u}$: $\Sigma_{y|u} = \Sigma_{yy} - \Sigma_{yu}\Sigma_{uu}^{-1}\Sigma_{uy}$

Expected value $m_{y|u}$: $m_{y|u} = m_y + \Sigma_{yu}\Sigma_{uu}^{-1}(u - m_u)$

Optimal model: $\bar{y} = \Phi^*(u) = \underset{\underline{y}}{E}\left[\underline{y}|\underline{u} = u\right] = m_{y|u} = m_y + \Sigma_{yu}\Sigma_{uu}^{-1}(u - m_u)$

$$\bar{y} = \Phi^*(u) = Au + b$$

where:

$$A = \Sigma_{yu}\Sigma_{uu}^{-1}$$

$$b = m_y - \Sigma_{yu}\Sigma_{uu}^{-1}m_u$$

Full a'priori knowledge

Model: $\bar{y} = \Phi(u,\theta)$

Φ – a given function, θ – vector of model parameters

$u \in \mathscr{U} \subseteq \mathscr{R}^S, \quad \bar{y} \in \overline{\mathscr{Y}} \subseteq \mathscr{R}^L, \quad \theta \in \Theta \subseteq \mathscr{R}^R$

Performance index: $Q(\theta) = \underset{\underline{u},\underline{y}}{E}\left[q\left(\underline{y},\Phi\left(\underline{u},\theta\right)\right)\right] = \int\limits_{\mathscr{U}} \int\limits_{\mathscr{Y}} q\left(y,\Phi\left(u,\theta\right)\right) f\left(u,y\right) dy\, du$

$$\theta^* \rightarrow \quad Q\left(\theta^*\right) = \min_{\theta \in \Theta} Q\left(\theta\right)$$

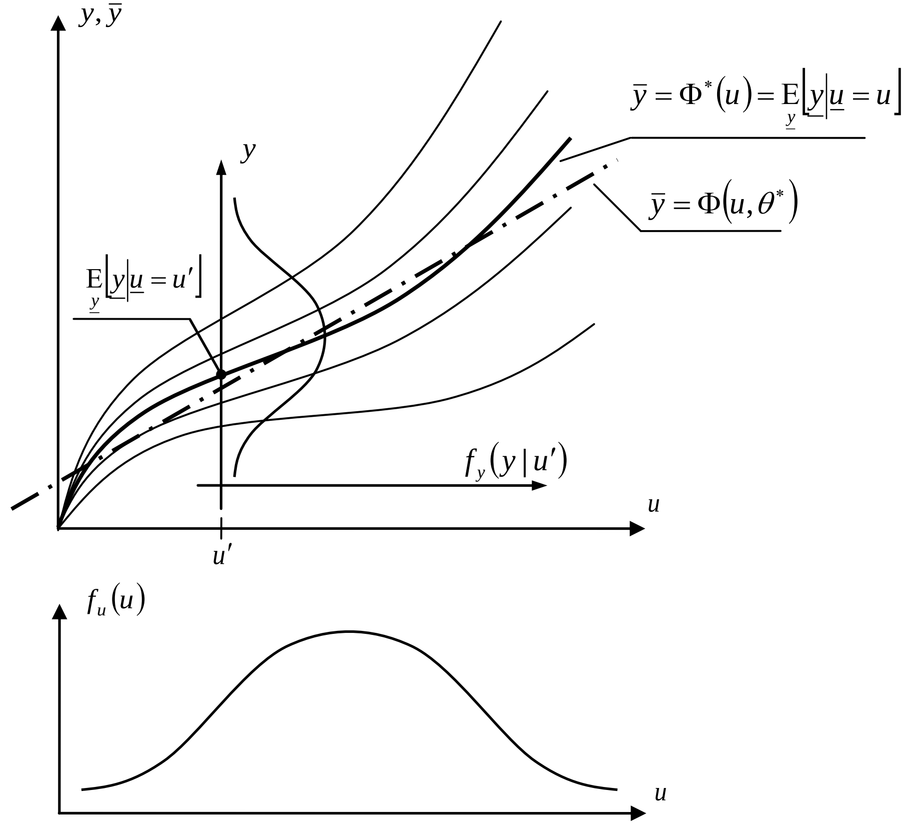
Optimal model: $\bar{y} = \Phi(u,\theta^*)$

Full a'priori knowledge

- Regression of the II type

$$q(y,\bar{y}) = [y-\bar{y}]^T [y-\bar{y}]$$

$$Q(\theta) = \int_{\mathcal{U}} \int_{\mathcal{Y}} [y-\Phi(u,\theta)]^T [y-\Phi(u,\theta)] \times$$
$$\times f(u,y)\,dy\,du$$



Full a'priori knowledge

Is there any relation between the I and the II type regression ?

$$\Phi^*(u) \sim ? \sim \Phi(u, \theta^*)$$

Performance index:

$$Q(\theta) = \underset{\underline{u},\underline{y}}{E}\left[\left(\underline{y} - \Phi(\underline{u}, \theta)\right)^2\right] = \int_{\mathcal{U}} \int_{\mathcal{Y}} \left(\underline{y} - \Phi(\underline{u}, \theta)\right)^2 f(u, y)\, dy\, du$$

Taking: $\left(\Phi^*(u) - \Phi^*(u) = 0\right)$ and $f(u, y) = f_y(y \mid u)\, f_u(u)$ we have:

$$Q(\theta) = \int_{\mathcal{U}} \int_{\mathcal{Y}} \left(\left(y - \Phi^*(u)\right) + \left(\Phi^*(u) - \Phi(u, \theta)\right)\right)^2 f_y(y|u) f_u(u)\, dy\, du$$

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Full a'priori knowledge

$$Q(\theta) = \int_{\mathcal{U}} \int_{\mathcal{Y}} \left(y - \Phi^*(u)\right)^2 f_y(y|u) f_u(u)\, dy\, du +$$

$$+ 2 \int_{\mathcal{U}} \left[\int_{\mathcal{Y}} \left(y - \Phi^*(u)\right)\left(\Phi^*(u) - \Phi(u,\theta)\right) f_y(y|u)\, dy \right] f_u(u)\, du +$$

$$+ \int_{\mathcal{U}} \int_{\mathcal{Y}} \left(\Phi^*(u) - \Phi(u,\theta)\right)^2 f_y(y|u) f_u(u)\, dy\, du =$$

$$= \int_{\mathcal{U}} Q_u\left(\Phi^*(u)\right) f_u(u)\, du +$$

$$+ 2 \int_{\mathcal{U}} \int_{\mathcal{Y}} \left(y - \Phi^*(u)\right) f_y(y|u)\, dy \left(\Phi^*(u) - \Phi(u,\theta)\right) f_u(u)\, du +$$

$$+ \int_{\mathcal{U}} \left(\Phi^*(u) - \Phi(u,\theta)\right)^2 \int_{\mathcal{Y}} f_y(y|u)\, dy\, f_u(u)\, du$$

Full a'priori knowledge

$$Q(\theta) = \sigma_y^2 +$$

$$+ 2 \int_{\mathscr{U}} \left[\int_{\mathscr{Y}} y f_y(y|u) dy - \Phi^*(u) \int_{\mathscr{Y}} f_y(y|u) dy \right] \left(\Phi^*(u) - \Phi(u,\theta) \right) f_u(u) du +$$

$$+ \int_{\mathscr{U}} \left(\Phi^*(u) - \Phi(u,\theta) \right)^2 f_u(u)\, du =$$

$$= \sigma_y^2 + 2 \int_{\mathscr{U}} \left[\Phi^*(u) - \Phi^*(u) \cdot 1 \right] \left(\Phi^*(u) - \Phi(u,\theta) \right) f_u(u) du +$$

$$+ \int_{\mathscr{U}} \left(\Phi^*(u) - \Phi(u,\theta) \right)^2 f_u(u)\, du =$$

$$= \sigma_y^2 + \int_{\mathscr{U}} \left(\Phi^*(u) - \Phi(u,\theta) \right)^2 f_u(u)\, du$$

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Full a'priori knowledge

$$Q(\theta) = \int_{\mathcal{U}} \int_{\mathcal{Y}} \left(y - \Phi^*(u)\right)^2 f(u, y)\, dy\, du + \int_{\mathcal{U}} \left(\Phi^*(u) - \Phi(u, \theta)\right)^2 f_u(u)\, du$$

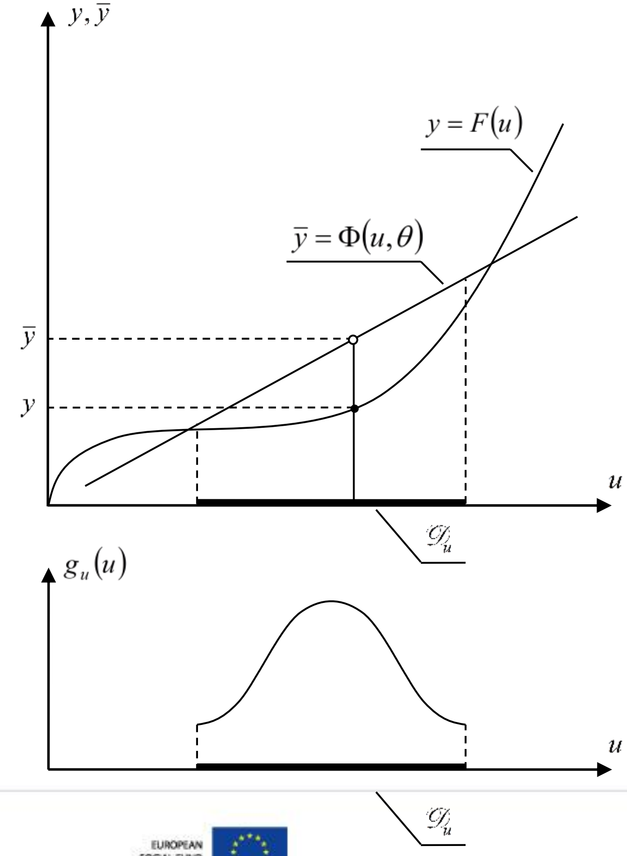
$$\theta^* \to \min_{\theta} Q(\theta) = \min_{\theta} \int_{\mathcal{U}} \left(\Phi^*(u) - \Phi(u, \theta)\right)^2 f_u(u)\, du$$

the I type regression — weight function

The II type regression is the best approximation of the I type regression.

Approximation of static plant characteristic

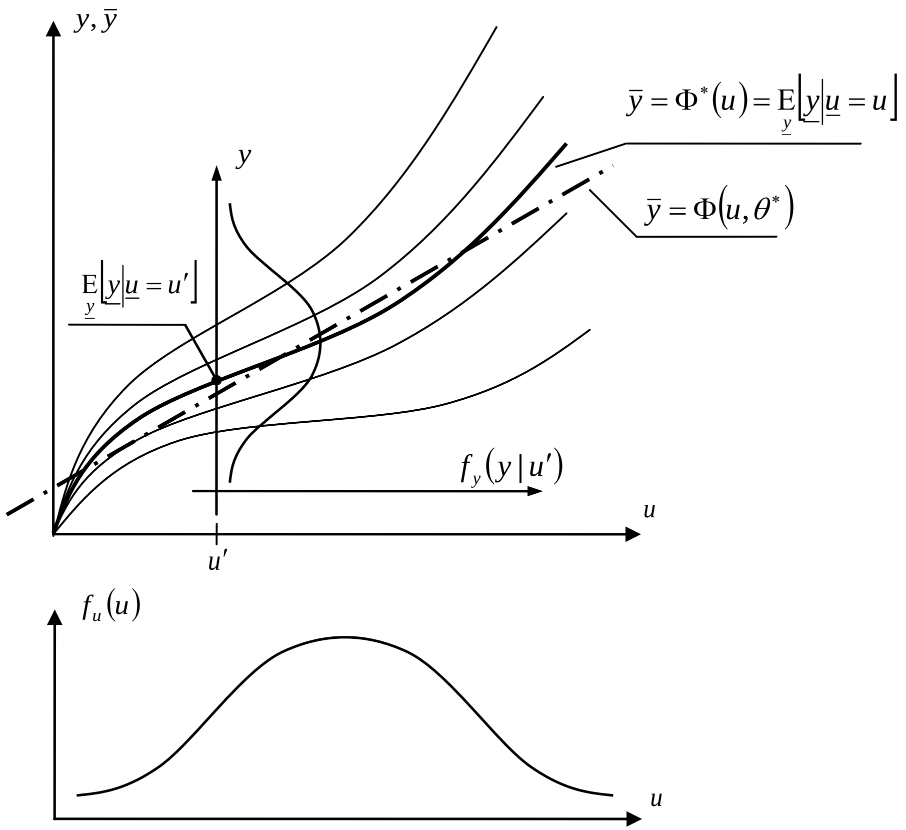
- Problem formulation



Weight function: $g_u(u)$

Full a’priori knowledge

- Relation between regression of the I and the II type



Choice of the best model, probabilistic case

Two possible cases

Full a'priori knowledge

- joint probability density function $f(u,y)$ of random variables $(\underline{u},\underline{y})$

or

- conditional probability density function $f_y(y|u)$

and mrginal probability density function

$$f_u(u)$$

are known

Incomplete probabilistic information

joint probability density function of random variables $(\underline{u},\underline{y})$ exist, but is not known.
Measurements:

$$(u_n, y_n),\ n=1,2,\ldots,N$$

are values of $(\underline{u},\underline{y})$

Unknown a Priori Knowledge

Joint probability density function of random variables $(\underline{u}, \underline{y})$ exists but is not known.

Measurements:

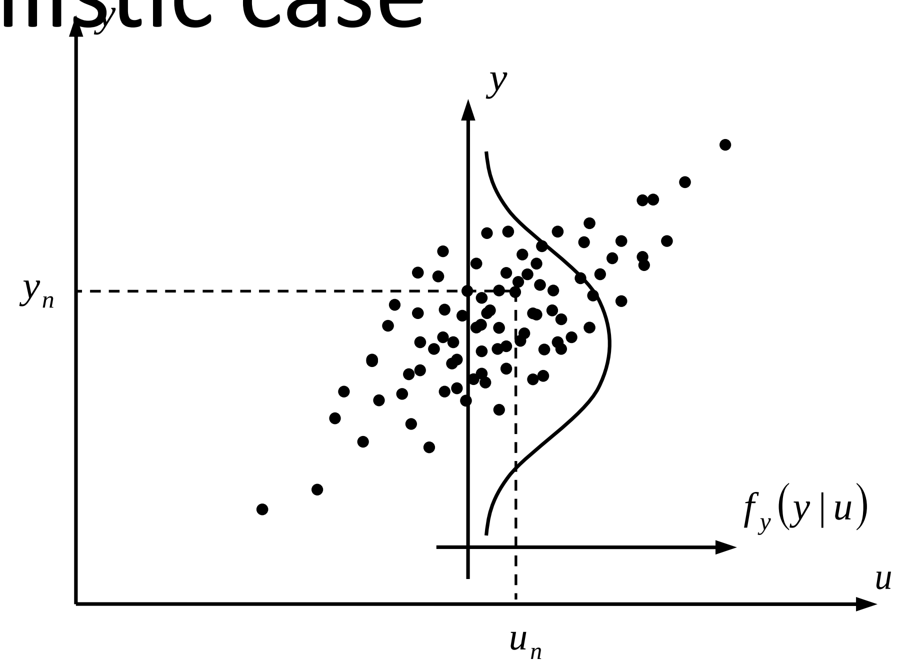
$$(u_n, y_n),\ n = 1, 2, \ldots, N \text{ are values of } (\underline{u}, \underline{y})$$

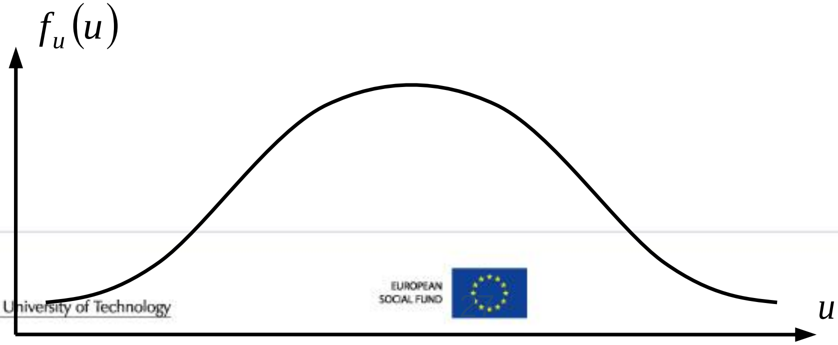
Choice of the best model, probabilistic case

$$\left(u_n, y_n\right), n = 1, 2, \ldots, N$$

are values of random variables

$$\left(\underline{u}, \underline{y}\right)$$





Unknown a Priori Knowledge

Empirical Estimation of the Performance Index

Empirical Estimation of the Performance Index

Empirical Probability Density Functions

Unknown parameters of the probability density functions

Empirical Probability Density Functions

Non parametric – Parzen estimation

Unknown a Priori Knowledge

Empirical Estimation of the Performance Index

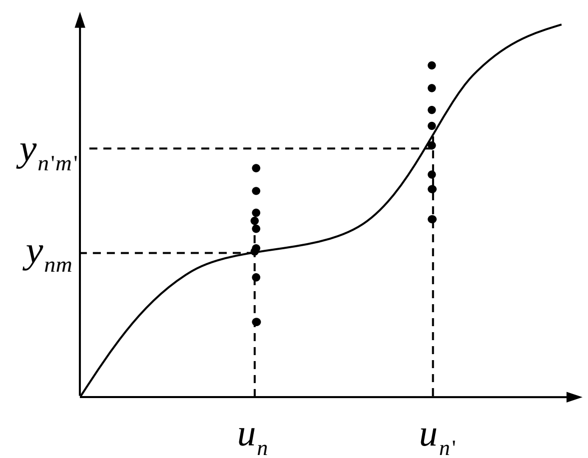
Empirical Estimation of the Performance Index

Experiment:

For the given: u_n output measurement are repeated y_{nm}.

Consequently:

$$(u_n, y_{nm}),\ m = 1, 2, \ldots, M_n,\ n = 1, 2, \ldots, \overline{N}$$



where:

M_n – number of output measurements for a given u_n.

$\overline{N}$ – number of the different input values

$N = \sum_{n=1}^{\overline{N}} M_n$ – number of measurements

Unknown a Priori Knowledge
Empirical Estimation of the Performance Index

Estimation of the performance index:

$$Q_u(\Phi(u)) = \underset{\underline{y}}{E}\left[q\left(\underline{y},\Phi(\underline{u})\right) \mid \underline{u}=u\right] = \int_{\mathcal{Y}} q(y,\Phi(u)) f_y(y \mid u)\,dy$$

for $u = u_n$

$$Q_{uM_n}(\Phi(u_n)) = \frac{1}{M_n}\sum_{m=1}^{M_n} q\left(y_{nm},\Phi(u_n)\right)$$

$$\Phi^*_{M_n}(u_n) \rightarrow \quad Q_{uM_n}\left(\Phi^*_{M_n}(u_n)\right) = \min_{\Phi(u_n)} Q_{uM_n}(\Phi(u_n))$$

$$\Phi^*_{M_n}(u_n), \quad n = 1, 2, \ldots, \overline{N}$$

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Unknown a Priori Knowledge

Empirical Estimation of the Performance Index

For

$$q(y,\bar{y}) = [y-\bar{y}]^T[y-\bar{y}]$$

Performance index:

$$Q_u(\Phi(u)) = \underset{\underline{y}}{E}\left[[y-\Phi(u)]^T[y-\Phi(u)] \middle| \underline{u}=u\right] = \int\limits_{\mathcal{Y}} [y-\Phi(u)]^T[y-\Phi(u)] f_y(y|u)\,dy$$

$$\bar{y} = \Phi^*(u) = \underset{\underline{y}}{E}\left[\underline{y} \middle| \underline{u}=u\right] = \int\limits_{\mathcal{Y}} y\, f(y|u)\,dy$$

Estimate of I type regression:

$$\bar{y}_n = \frac{1}{M_n}\sum_{m=1}^{M_n} y_{nm},\ \ n = 1, 2, \ldots \overline{N}$$

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Unknown a Priori Knowledge
Empirical Estimation of the Performance Index

Empirical estimation of the performance index:

$$Q(\theta) = \underset{\underline{u},\underline{y}}{E}\left[(y - \Phi(u,\theta))^2\right] = \iint_{\mathcal{U}\mathcal{Y}} (y - \Phi(u,\theta))^2 f(u,y)dydu$$

or

$$Q_N(\theta) = \frac{1}{N}\sum_{n=1}^{N} q(y_n, \Phi(u_n\theta))$$

$$Q_N(\theta) = \frac{1}{N}\sum_{n=1}^{\bar{N}}\sum_{m=1}^{M_n} q(y_{nm}, \Phi(u_n, \theta))$$

$$\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta\in\Theta} Q_N(\theta)$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

Empirical Probability Density Functions

Unknown parameters of the probability density functions

$$f(u,y) = f_{\alpha}(u,y,\alpha)$$

f_{α} known function, $\alpha \in \mathcal{A}$ unknown function parameters

Likelihood function

$$L(U_N, Y_N, \alpha) = \prod_{n=1}^{N} f_{\alpha}(u_n, y_n, \alpha)$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

Estimate:

$$\alpha_N = \Psi_N(U_N, Y_N) \quad \rightarrow \quad L(U_N, Y_N, \alpha_N) = \max_{\alpha \in \mathcal{A}} L(U_N, Y_N, \alpha)$$

$$Q_{u\alpha_N}(\Phi(u), \alpha_N) = \int_{\mathcal{Y}} q(y, \Phi(u)) \frac{f_\alpha(u, y, \alpha_N)}{\int_{\mathcal{Y}} f_\alpha(u, y, \alpha_N) dy} dy$$

$$\Phi^*_{\alpha_N}(u) \rightarrow Q_{u\alpha_N}(\Phi^*_{\alpha_N}(u), \alpha_N) = \min_{\Phi(u)} Q_{u\alpha_N}(\Phi(u), \alpha_N)$$

$$\bar{y} = \Phi^*_{\alpha_N}(u, \alpha_N) = \int_{\mathcal{Y}} y \frac{f_\alpha(u, y, \alpha_N)}{\int_{\mathcal{Y}} f_\alpha(u, y, \alpha_N) dy} dy$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

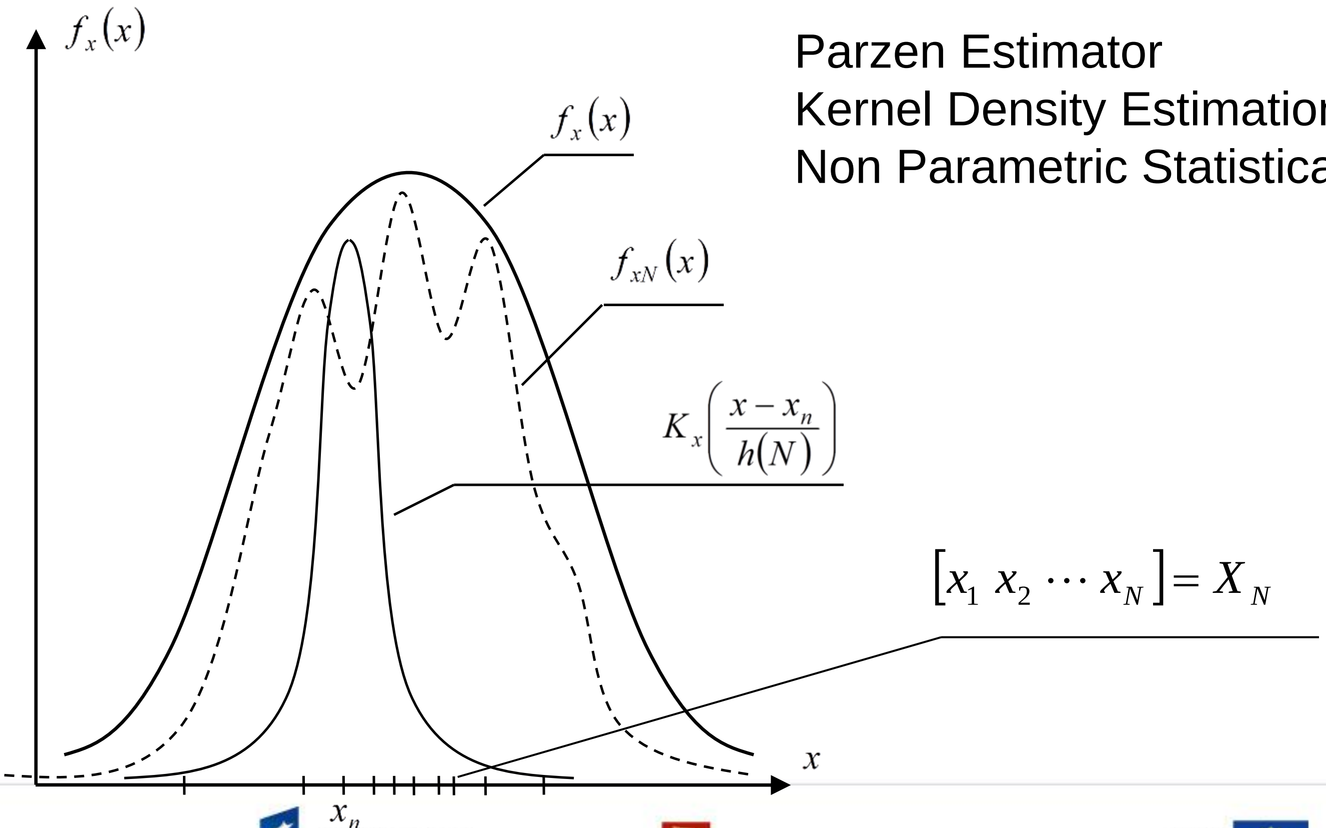
Estimate (continued):

For
$$q(y,\bar{y}) = [y-\bar{y}]^T [y-\bar{y}]$$

$$Q_{\alpha_N}(\theta,\alpha_N) = \int_{\mathcal{U}}\int_{\mathcal{Y}} q(y,\Phi(u,\theta))\, f_\alpha(u,y,\alpha_N)\,dy\,du$$

$$\theta^*_{\alpha_N} \rightarrow \quad Q_{\alpha_N}\left(\theta^*_{\alpha_N},\alpha_N\right) = \min_{\theta\in\Theta} Q_{\alpha_N}(\theta,\alpha_N)$$

Unknown a Priori Knowledge
Empirical Probability Density Functions



Parzen Estimator
Kernel Density Estimation
Non Parametric Statistical Estimation

Unknown a Priori Knowledge
Empirical Probability Density Functions

$\underline{x}$ – random variable, probability density function $f_x(x)$ exists but is unknown.

Observations:

$$[x_1 \; x_2 \; \cdots \; x_N] = X_N$$

Parzen's estimate:

$$f_x(x) \approx f_{xN}(x; X_N) = \frac{1}{N h^P(N)} \sum_{n=1}^{N} K_x\left(\frac{x - x_n}{h(N)}\right)$$

where:

$$h(N) > 0, \; \lim_{N\to\infty} h(N) = 0, \; \lim_{N\to\infty} h^P(N) = \infty$$

K_x – known function (kernel), such that:

$$\sup_{x\in\mathcal{X}} |K_x(x)| < \infty, \; \int_{\mathcal{X}} K_x(x)\,dx = 1, \; \int_{\mathcal{X}} |K_x(x)|\,dx < \infty, \; \lim_{\|x\|\to\infty} |K_x(x)| = 0$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

Example

$$h(N) = cN^{-\alpha}$$

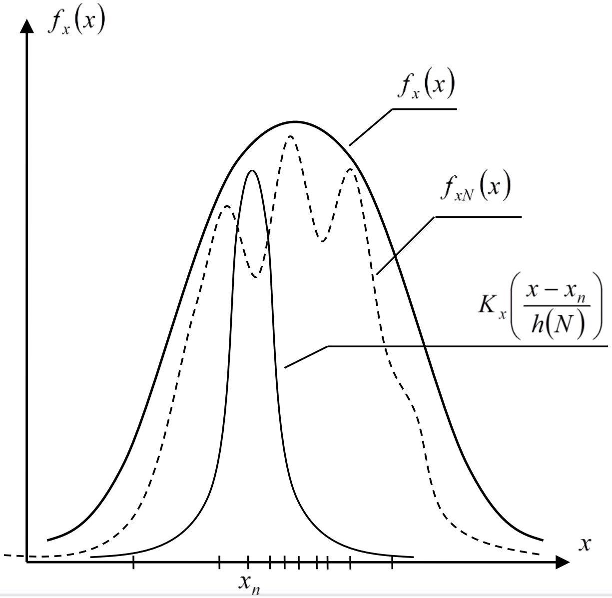
where: c and α are numbers;

$$c > 0,\ 0 < \alpha < \frac{1}{P}$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

Examples

$$K_x(x)=\begin{cases}1\ dla\ \|x\|\le 1\\ 0\ dla\ \|x\|>1\end{cases},\qquad K_x(x)=(2\pi)^{-\frac{P}{2}}e^{-\frac{1}{2}\|x\|^2},\quad K_x(x)=\pi^{-P}\prod_{p=1}^{P}\sqrt{1+\left|x^{(p)}\right|}$$



Unknown a Priori Knowledge
Empirical Probability Density Functions

Joint probability density function of random variables $(\underline{u}, \underline{y})$, $f(u, y)$

$$f_N(u, y; U_N, Y_N) = \frac{1}{N\, h^{L+S}(N)} \sum_{n=1}^{N} K_u\left(\frac{u - u_n}{h(N)}\right) K_y\left(\frac{y - y_n}{h(N)}\right)$$

Probability density function $f_u(u)$:

$$f_{uN}(u; U_N) = \frac{1}{N\, h^{S}(N)} \sum_{n=1}^{N} K_u\left(\frac{u - u_n}{h(N)}\right)$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

and conditional probability $f_y(y|u)$:

$$f_{yN}(y|u;U_N,Y_N)=\frac{f_N(u,y;U_N,Y_N)}{f_{uN}(u;U_N)}=$$

$$=\frac{1}{h^L(N)}\frac{\sum_{n=1}^{N}K_u\left(\frac{u-u_n}{h(N)}\right)K_y\left(\frac{y-y_n}{h(N)}\right)}{\sum_{n=1}^{N}K_u\left(\frac{u-u_n}{h(N)}\right)}$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

Consequently:

$$Q_{uN}\left(\Phi(u);U_N,Y_N\right)=\int_{\mathcal{Y}} q\left(y,\Phi(u)\right)f_{yN}\left(y|u;U_N,Y_N\right)dy$$

$$\bar{y}=\Phi_N^*\left(u;U_N,Y_N\right)\rightarrow Q_{uN}\left(\Phi_N^*\left(u,U_N,Y_N\right),U_N,Y_N\right)=\min_{\Phi(u)} Q_{uN}\left(\Phi(u),U_N,Y_N\right)$$

For:

$$q(y,\bar{y})=[y-\bar{y}]^T[y-\bar{y}]$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

First type regression

$$\bar{y} = \Phi_N^*(u; U_N, Y_N) = \int_{\mathcal{Y}} y \; f_{yN}(y|u; U_N, Y_N)\, dy =$$

$$= \frac{1}{h^L(N)} \frac{\sum_{n=1}^{N} K_u\left(\frac{u-u_n}{h(N)}\right) \int_{\mathcal{Y}} y\, K_y\left(\frac{y-y_n}{h(N)}\right) dy}{\sum_{n=1}^{N} K_u\left(\frac{u-u_n}{h(N)}\right)}$$

Under condition for $K_y(y)$: $K_y(y) = K_y(-y)$ we have:

$$\frac{1}{h^L(N)} \int_{\mathcal{Y}} y\, K_y\left(\frac{y-y_n}{h(N)}\right) dy = y_n$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

and first type regression:

$$\bar{y} = \Phi_N^*\left(u; U_N, Y_N\right) = \frac{\sum_{n=1}^{N} y_n K_u\left(\frac{u-u_n}{h(N)}\right)}{\sum_{n=1}^{N} K_u\left(\frac{u-u_n}{h(N)}\right)}$$

Unknown a Priori Knowledge
Empirical Probability Density Functions

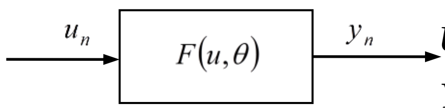
Optimal model:

$$Q_N(\theta; U_N, Y_N) = \int_{\mathcal{U}} \int_{\mathcal{Y}} q(y, \Phi(u,\theta)) f_N(u, y; U_N, Y_N)\, dy\, du$$

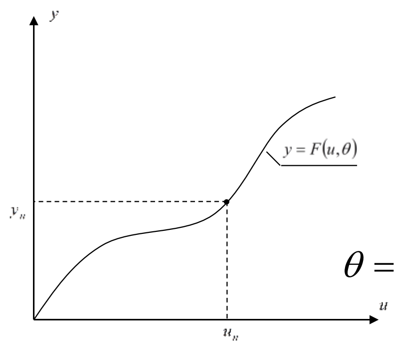
$$\theta_N^*(U_N, Y_N) \rightarrow \quad Q_N(\theta_N^*; U_N, Y_N) = \min_{\theta \in \Theta} Q_N(\theta; U_N, Y_N)$$

Plant in the class of model

Deterministyczny

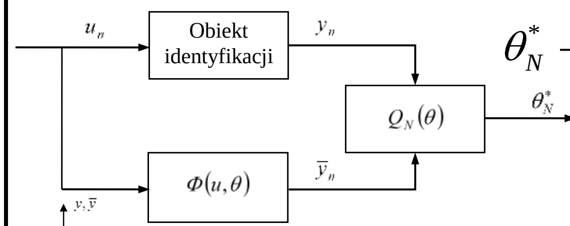


$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

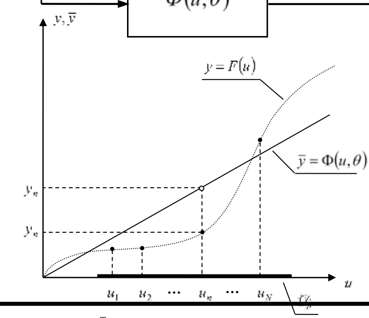


$$y_n = F(u_n, \theta),$$
$$n = 1, 2, \ldots, N$$
$$Y_N = \overline{F}(U_N, \theta)$$
$$\theta = \overline{F}^{-1}(U_N, Y_N) \overset{df}{=} \Psi_N(U_N, Y_N)$$

Choice of the best model



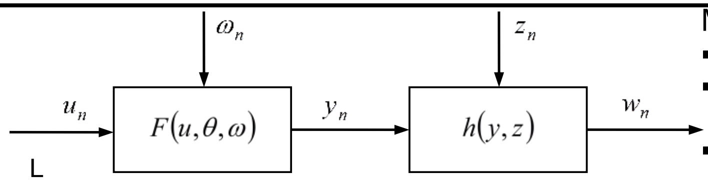
$$\overline{y} = \Phi(u, \theta_N^*)$$
$$\theta_N^* \rightarrow Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$$
$$\theta_N^* = \Psi_N(U_N, Y_N)$$
$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$



Optimal model for:

- Given measurements sequences
- Klass of models
- Given performance index

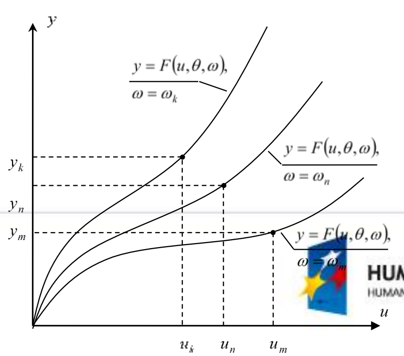
Losowy

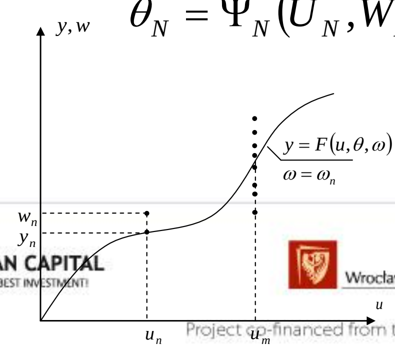


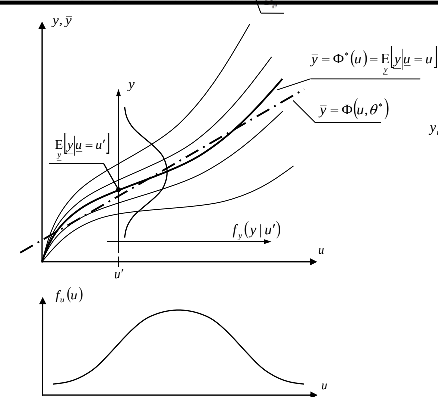
Methods:

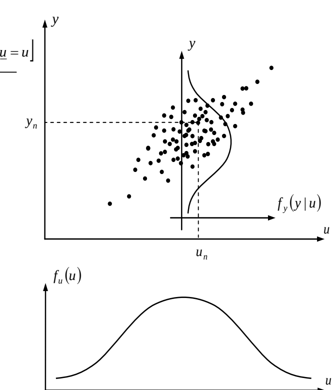
- Least square
- Maximum likelihood
- Bayes'a

$$\theta_N = \Psi_N(U_N, W_N)$$









Full probabilistic knowledge

- First type regrsesion
- Second type regression

Unknown probabilostic knowledge

- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

Thank you for attention