

Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

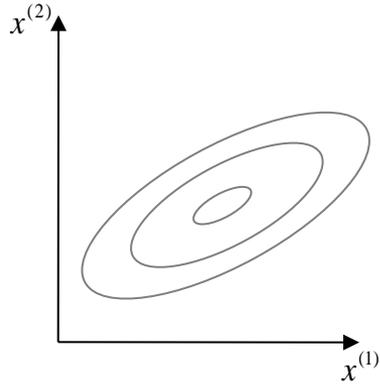
L.5a Numerical optimization methods – line search



Project co-financed from the EU European Social Fund

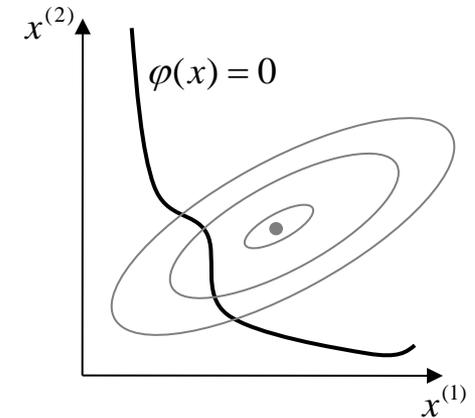


General classification of optimization tasks



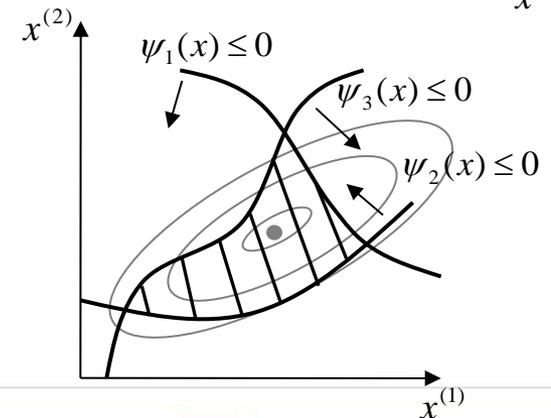
Unconstrained optimization:

$$\mathcal{D}_x = \mathcal{R}^S$$



Optimization under equality constraints:

$$\mathcal{D}_x = \{x \in \mathcal{R}^S : \varphi_1(x) = 0, \varphi_2(x) = 0, \dots, \varphi_L(x) = 0, L \leq S\}$$



Optimization under inequality constraints:

$$\mathcal{D}_x = \{x \in \mathcal{R}^S : \psi_1(x) \leq 0, \psi_2(x) \leq 0, \dots, \psi_M(x) \leq 0\}$$



Analytical methods

- Unconstrained optimization
- Lagrange multipliers method – equality constraints
- Kuhn-Tucker conditions – inequality constraints



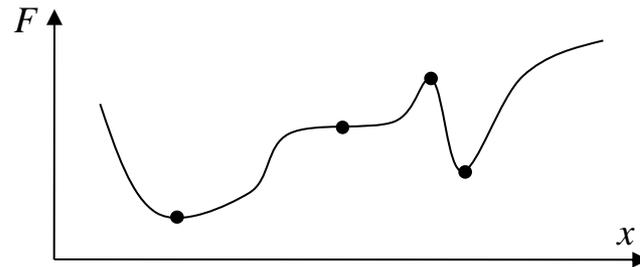
Unconstrained optimization

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x)$

Assumption: $F(x)$ is continuous and differentiable.

Necessary condition for x^* to be local minima: $\nabla_x F(x^*) = 0_S$

If $F(x)$ is convex function, then above equation is sufficient condition for x^* to be global minima.





Optimization under equality constraints

- The method of Lagrange multipliers

Lagrange function:

$$L(x, \lambda) = F(x) + \sum_{l=1}^L \lambda_l \varphi_l(x) = F(x) + \lambda^T \varphi(x)$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_L \end{bmatrix}, \quad \varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_L(x) \end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x, \lambda) \Big|_{x^*, \lambda^*} = 0_S$$

$$\nabla_\lambda L(x, \lambda) \Big|_{x^*, \lambda^*} = 0_L \quad \text{if and only if} \quad \text{rank } G(x) = \text{rank} \begin{bmatrix} G(x) & \vdots & -\nabla_x F(x) \end{bmatrix},$$

$$\text{Where: } G(x) = \begin{bmatrix} \nabla_x \varphi_1(x) & \vdots & \nabla_x \varphi_2(x) & \vdots & \dots & \vdots & \nabla_x \varphi_L(x) \end{bmatrix}$$



Optimization under inequality constraints

Lagrange function:

Kuhn-Tucker conditions

$$L(x, \mu) = F(x) + \mu^T \psi(x) \quad \Leftrightarrow \quad L(x, \mu) = F(x) + \sum_{m=1}^M \mu_m \psi_m(x)$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_M \end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x, \mu) \Big|_{x^*, \mu^*} = 0_S$$

$$\mu^T \nabla_\mu L(x, \mu) \Big|_{x^*, \mu^*} = 0$$

$$\nabla_\mu L(x, \mu) \Big|_{x^*, \mu^*} \leq 0_M$$

$$\mu^* \geq 0_M$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_S \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_S \end{bmatrix}$$

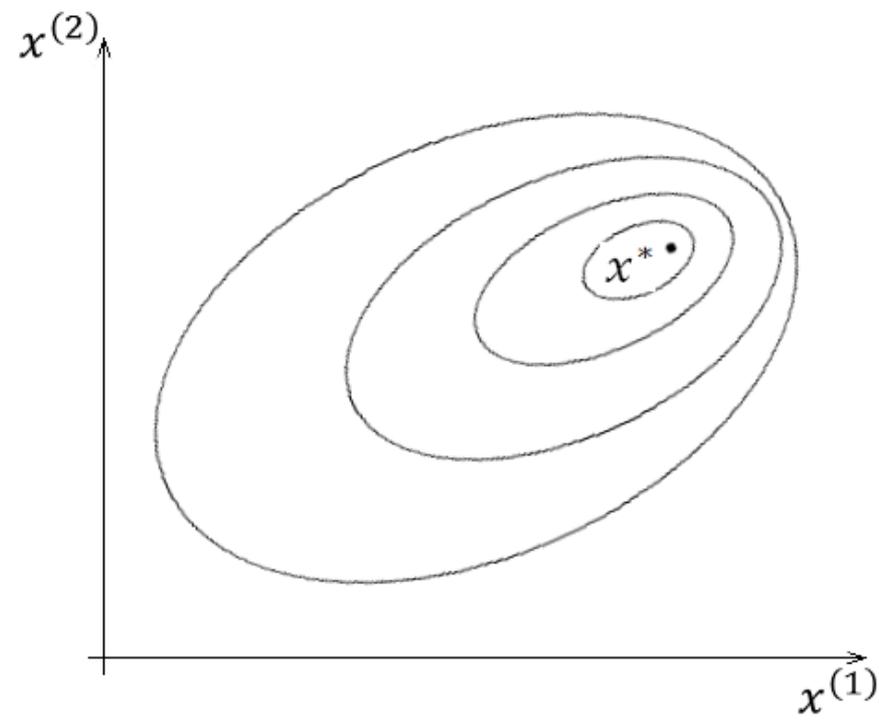
$$\alpha \leq \beta \Rightarrow \forall_{s=1, \dots, S} \alpha_s \leq \beta_s$$

If solution is regular



Numerical optimization methods

$$x^* \rightarrow F(x^*) = \min_{x \in D_x} F(x)$$



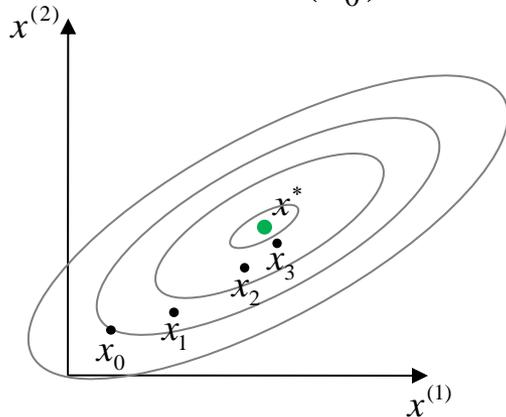
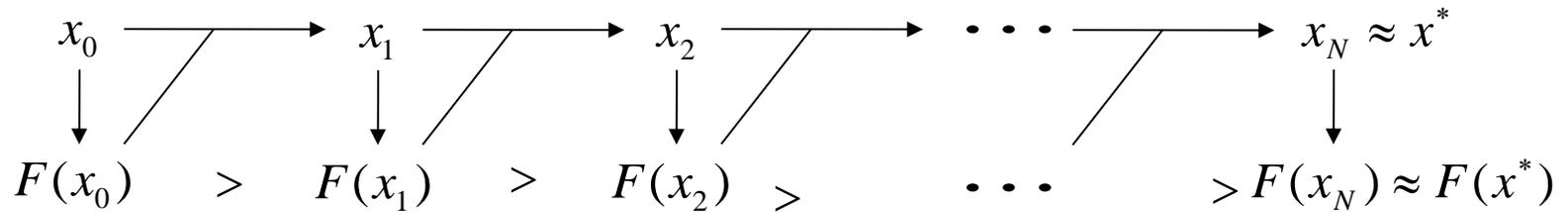
Analytical methods has drawbacks, when:

1. The goal function F and constraints φ, ψ are nonlinear.
2. Functions F, φ and ψ are non-differentiable
3. Mathematical formula describing functions F, φ and ψ is not available, it can only be „measured”
4. Large dimension of decision variables vector



Numerical methods

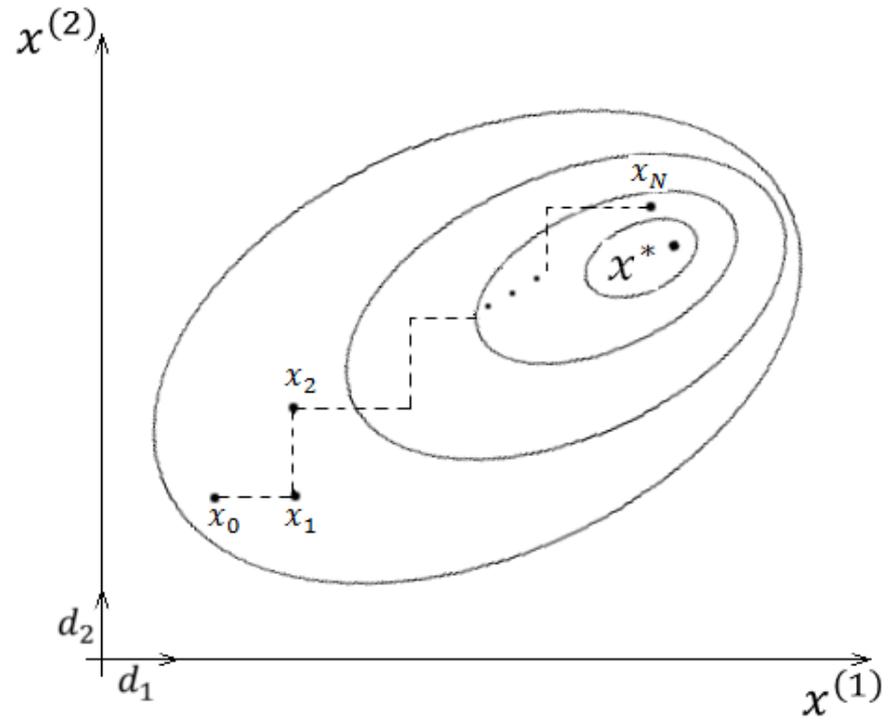
We only use information about values of objective function $F(x)$ for a given value of x .



The general idea behind numerical methods.



Numerical optimization methods



Algorithm

$$x_{n+1} = \Psi(x_n), x_0$$

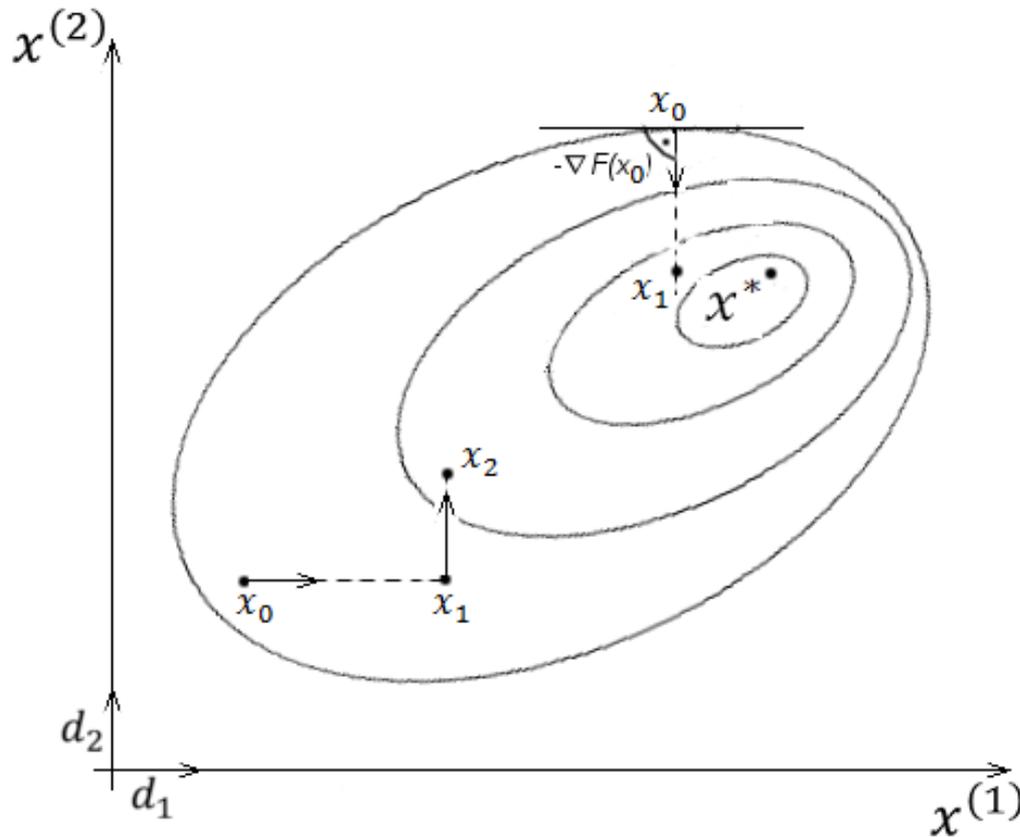
- Choice of the search direction.
- Line search optimization.
- Stopping conditions.

$$x_0, x_1, \dots, x_n, \dots, x_N \approx x^*$$

$$F(x_0) > F(x_1) > \dots > F(x_n) > \dots > F(x_N) \approx F(x^*)$$



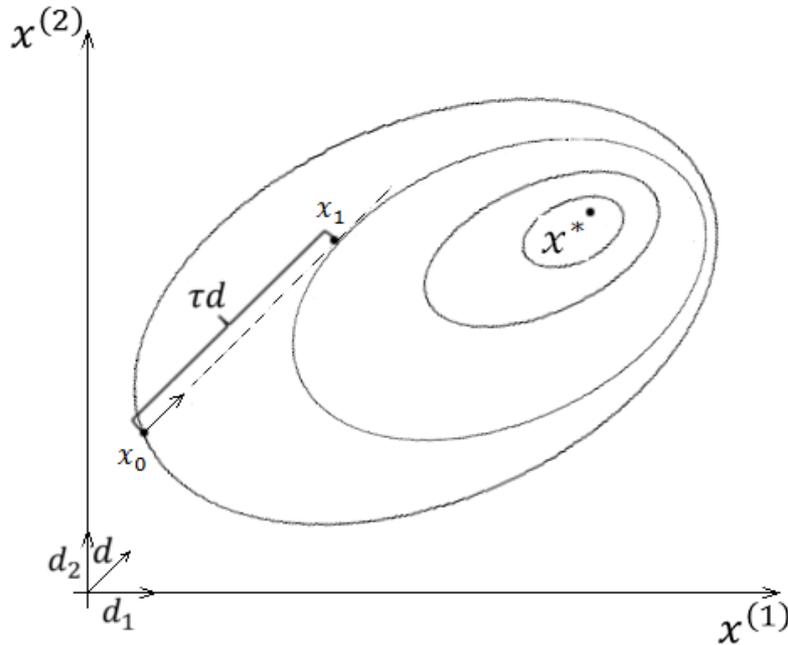
Choice of the search direction



- Basis of search directions – non-gradient methods.
- Search directions based on gradient vectors – gradient-based methods.



Line search optimization



x_0 – initial solution

x_1 – next solution

d – search direction

τ – step size

$$\tau^* \rightarrow F(x_0 + \tau^*d) = \min_{\tau} F(x_0 + \tau d)$$

x_0, d – fixed

$$F(x_0 + \tau d) \triangleq f(\tau)$$

$f(\tau)$ – a single variable function
(of the step size τ)

$$\tau^* \rightarrow f(\tau^*) = \min_{\tau} f(\tau)$$

line search optimization \equiv optimization of a single variable function

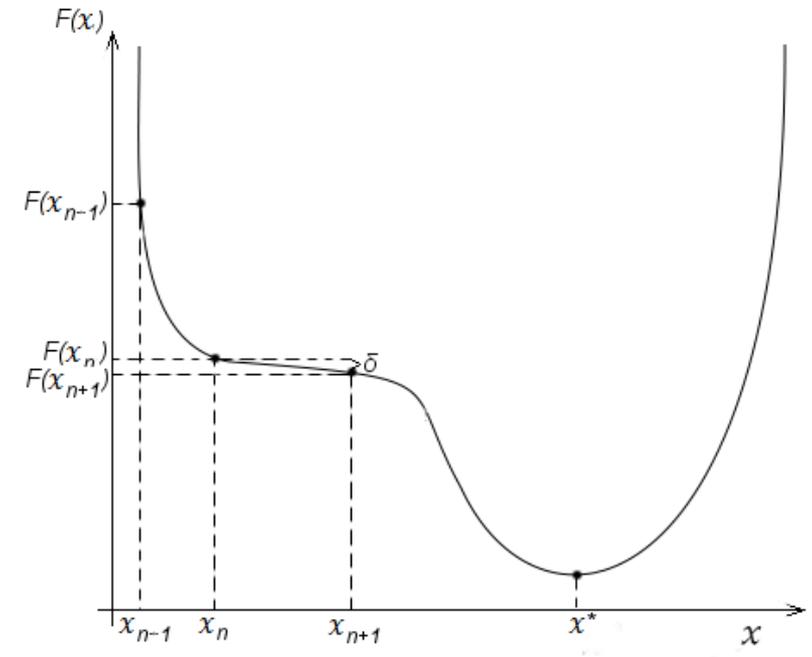
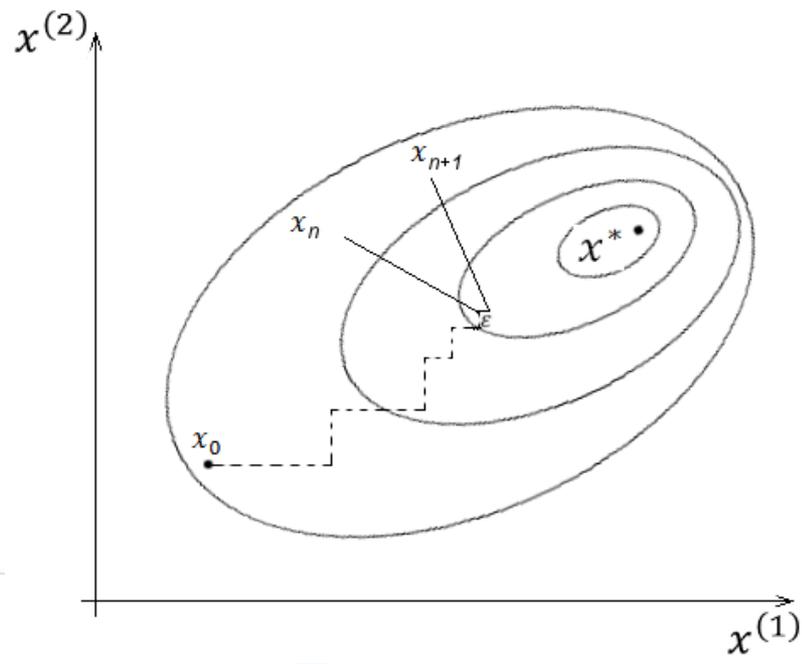


Stopping conditions

$$\|x_{n+1} - x_n\| < \varepsilon;$$

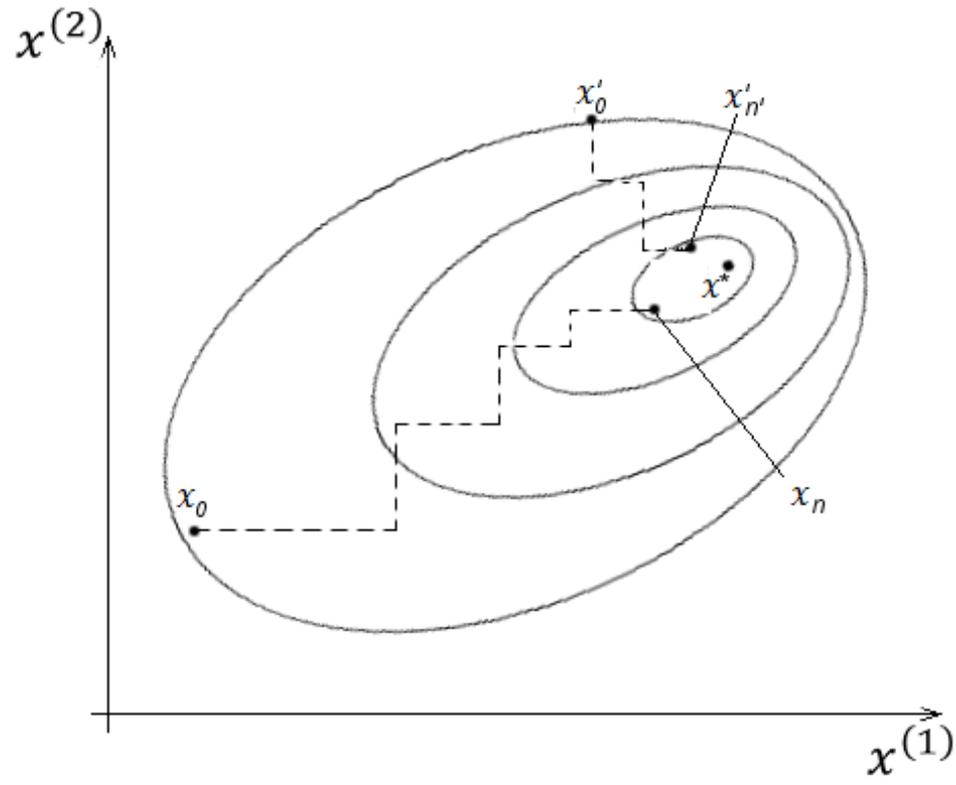
$$|F(x_{n+1}) - F(x_n)| < \delta;$$

$$\frac{|F(x_{n+1}) - F(x_n)|}{\|x_{n+1} - x_n\|} < \rho$$





Remedy

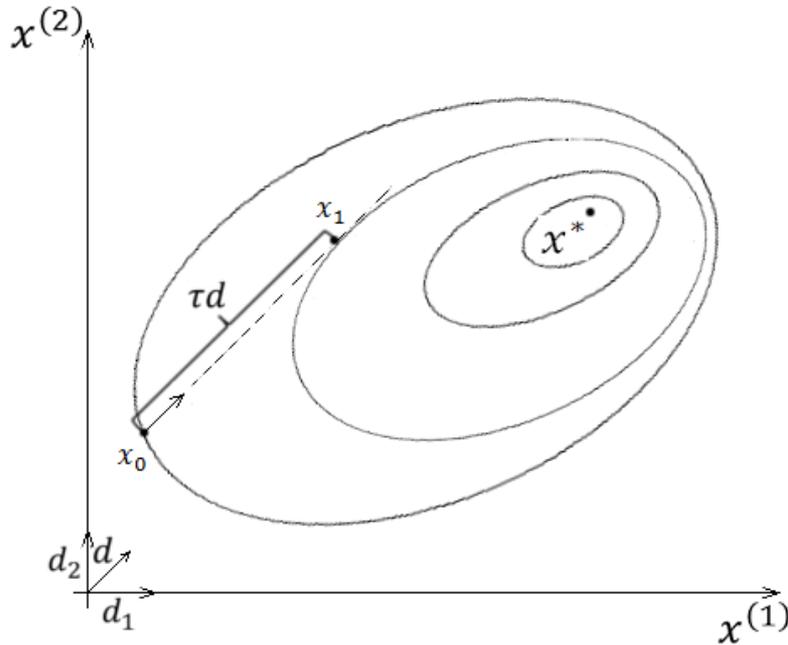


$$\|x_n - x'_{n'}\| < \varepsilon$$

x_0, x'_0 – different initial solutions
 $x_n, x'_{n'}$ – responding final solutions



Line search optimization



x_0 – initial solution

x_1 – next solution

d – search direction

τ – step size

$$\tau^* \rightarrow F(x_0 + \tau^*d) = \min_{\tau} F(x_0 + \tau d)$$

x_0, d – fixed

$$F(x_0 + \tau d) \triangleq f(\tau)$$

$f(\tau)$ – a single variable function
(of the step size τ)

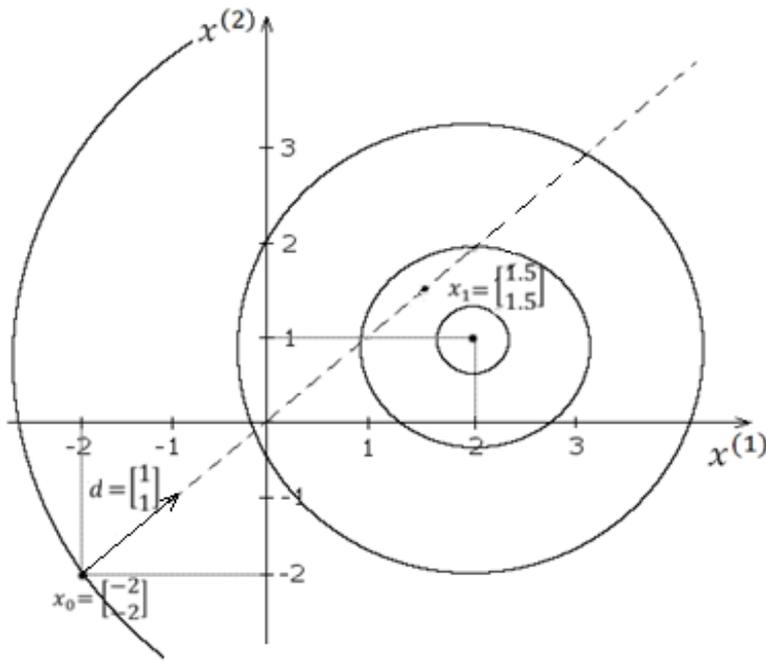
$$\tau^* \rightarrow f(\tau^*) = \min_{\tau} f(\tau)$$

line search optimization \equiv optimization of a single variable function



Example

$$F(x) = (x^{(1)} - 2)^2 + (x^{(2)} - 1)^2, \quad x_0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



$$x_1 = x_0 + \tau d = \begin{bmatrix} -2 \\ -2 \end{bmatrix} + \tau \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 + 1 * \tau \\ -2 + 1 * \tau \end{bmatrix}$$

$$F(x_0 + \tau d) = (-2 + \tau - 2)^2 + (-2 + \tau - 1)^2 = (\tau - 4)^2 + (\tau - 3)^2 = 2\tau^2 - 14\tau + 25 \triangleq f(\tau)$$

$$f'(\tau) = 4\tau - 14 = 0$$

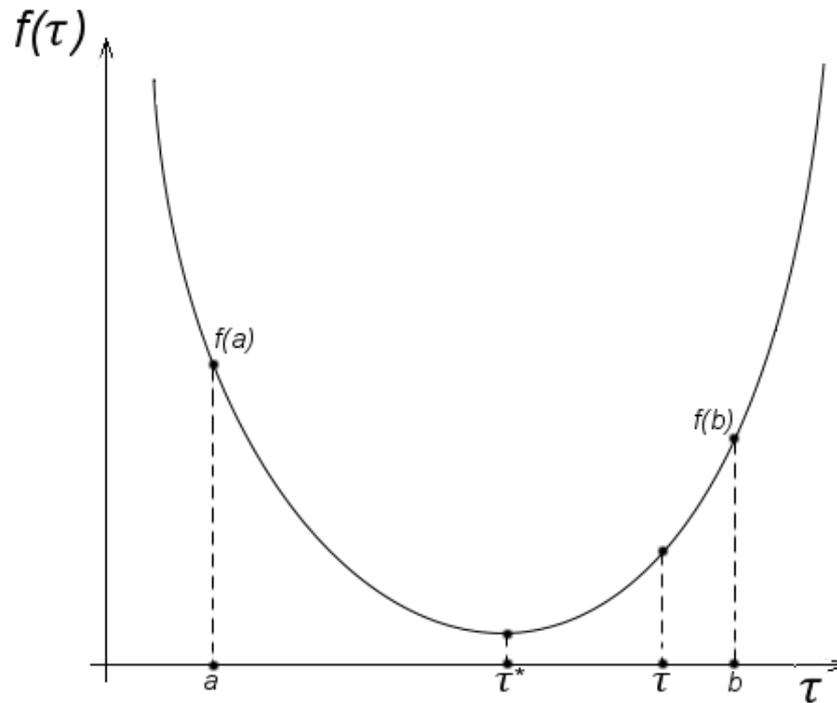
$$\tau^* = 3.5$$

$$x_1 = x_0 + \tau^* d = \begin{bmatrix} -2 \\ -2 \end{bmatrix} + 3.5 * \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$$



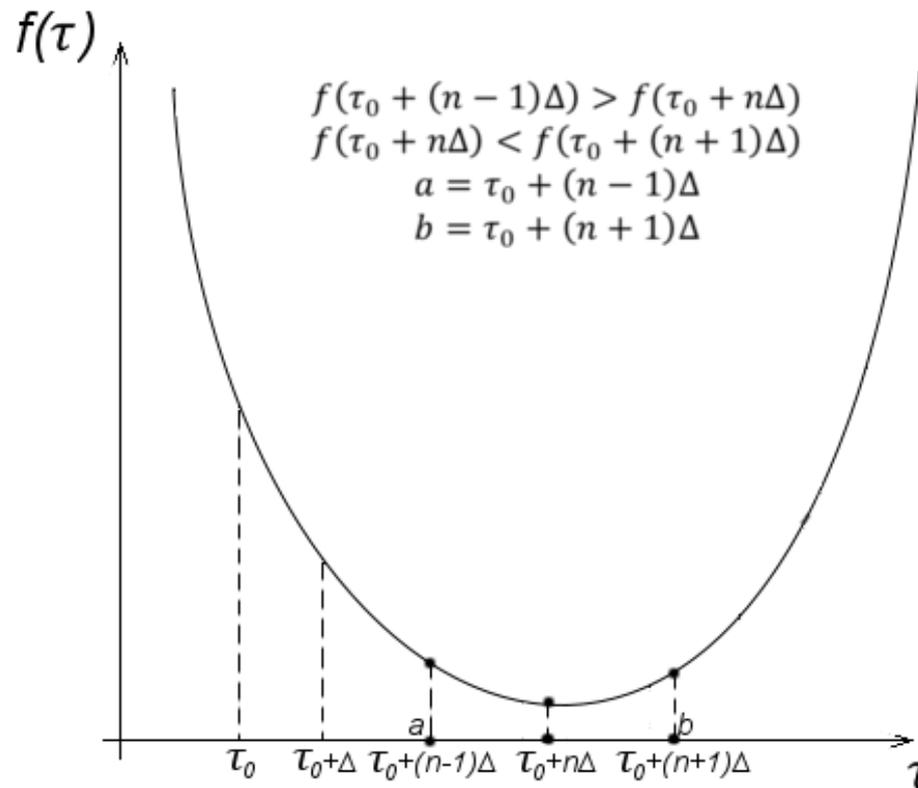
Reducing the interval of uncertainty

Assumption: $\tau^* \in [a, b]$



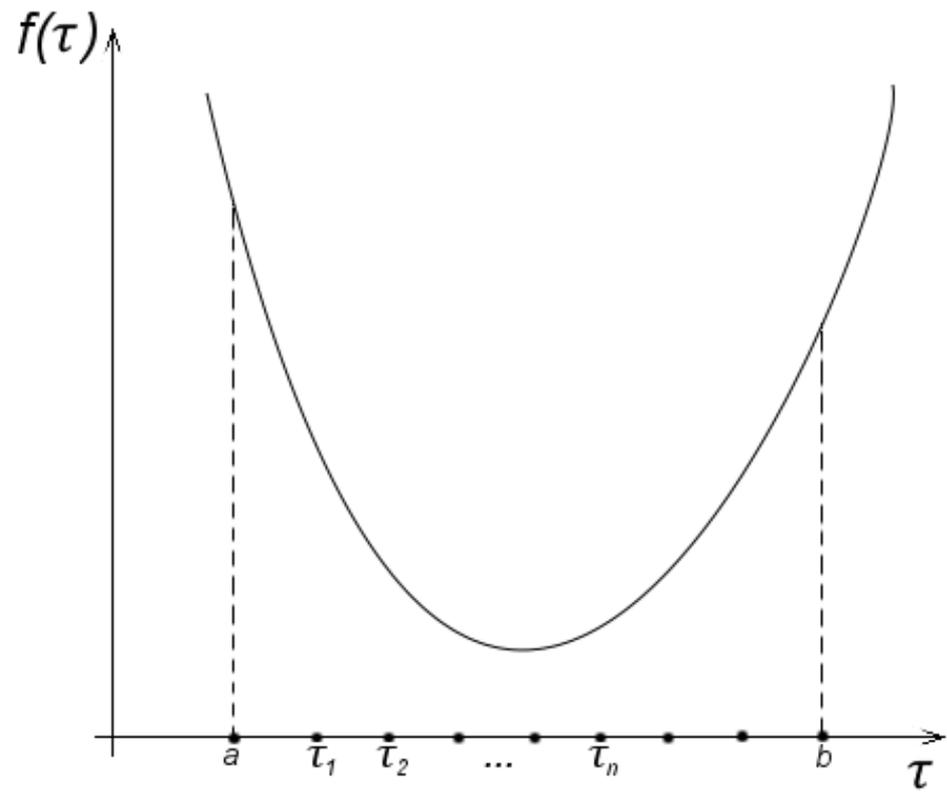


$$\tau \in [a, b]$$





Uniform search method



$N = \left\lceil \frac{b-a}{\varepsilon} \right\rceil$ – the total number of the goal function evaluations

For example:

$$\Delta = b - a = 1, \varepsilon = 0.01$$

$$N = \left\lceil \frac{b - a}{\varepsilon} \right\rceil = \frac{1}{0.01} = 100$$

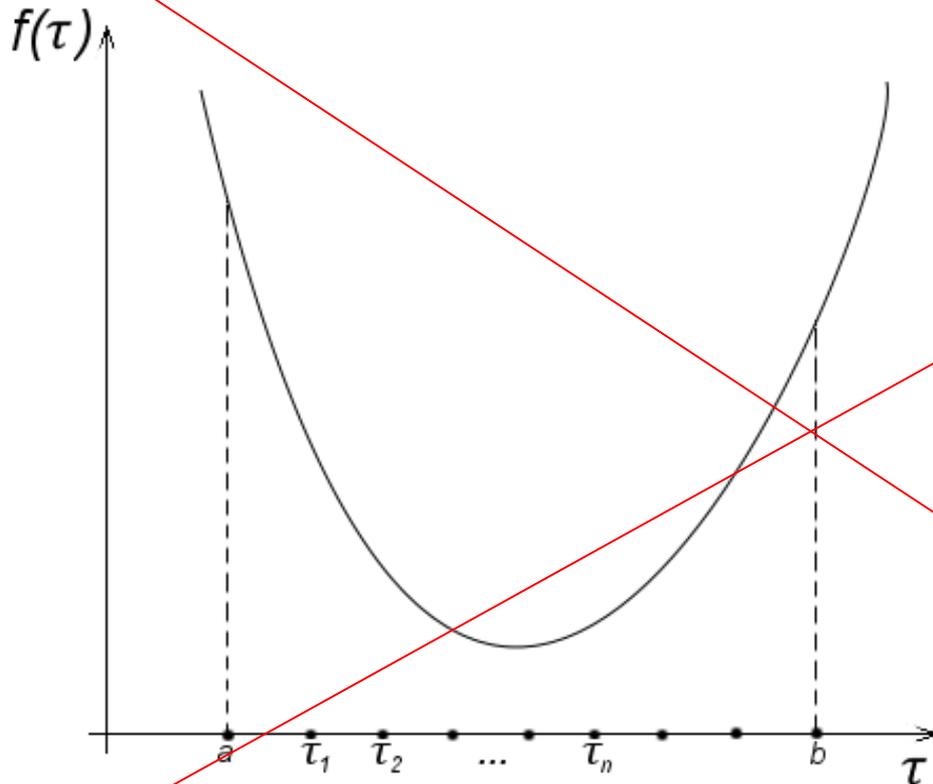
$$\tau_0 = a$$

$$\tau_n = \tau_0 + n\varepsilon$$

$$\tau^* \approx \tilde{\tau} \rightarrow f(\tilde{\tau}) = \min_{1 \leq n \leq N} \{f(\tau_n)\}$$



Uniform search method



$N = \left\lceil \frac{b-a}{\varepsilon} \right\rceil$ – the total number of the goal function evaluations

For example:

$$\Delta = b - a = 1, \varepsilon = 0.01$$

$$N = \left\lceil \frac{b - a}{\varepsilon} \right\rceil = \frac{1}{0.01} = 100$$

$$\tau_0 = a$$

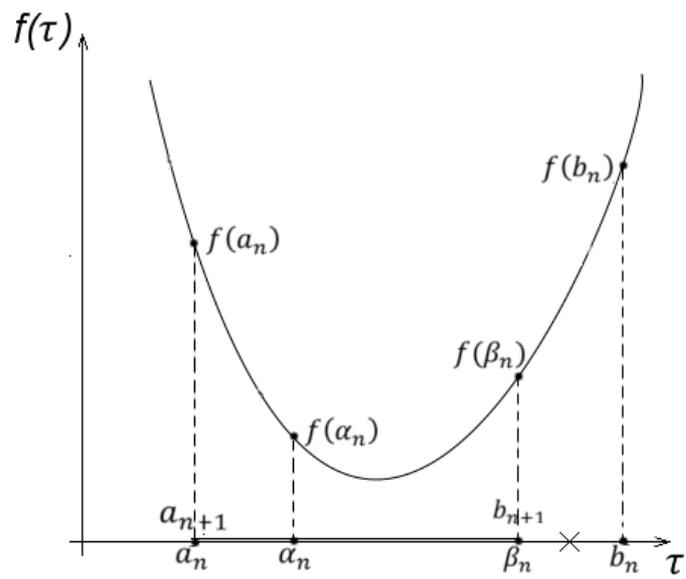
$$\tau_n = \tau_0 + n\varepsilon$$

$$\tau^* \approx \tilde{\tau} \rightarrow f(\tilde{\tau}) = \min_{1 \leq n \leq N} \{f(\tau_n)\}$$



Splitting the section into two parts

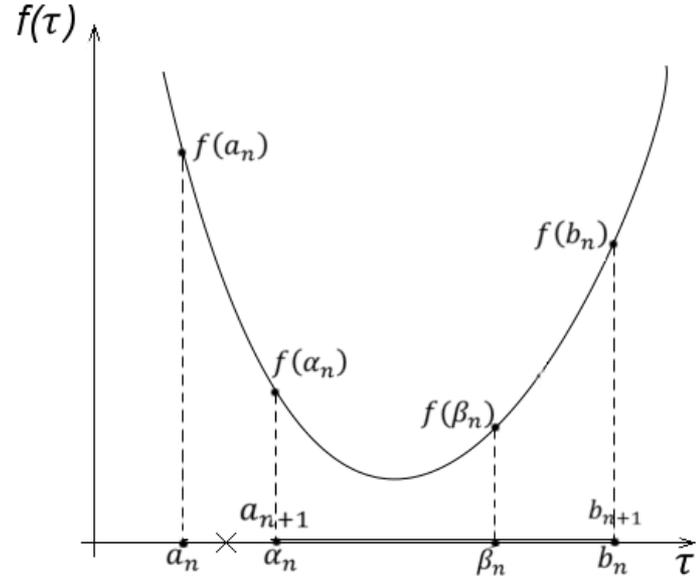
$f(\alpha_n)? f(\beta_n)$



$$f(\alpha_n) \leq f(\beta_n)$$

$$a_{n+1} := a_n$$

$$b_{n+1} := \beta_n$$



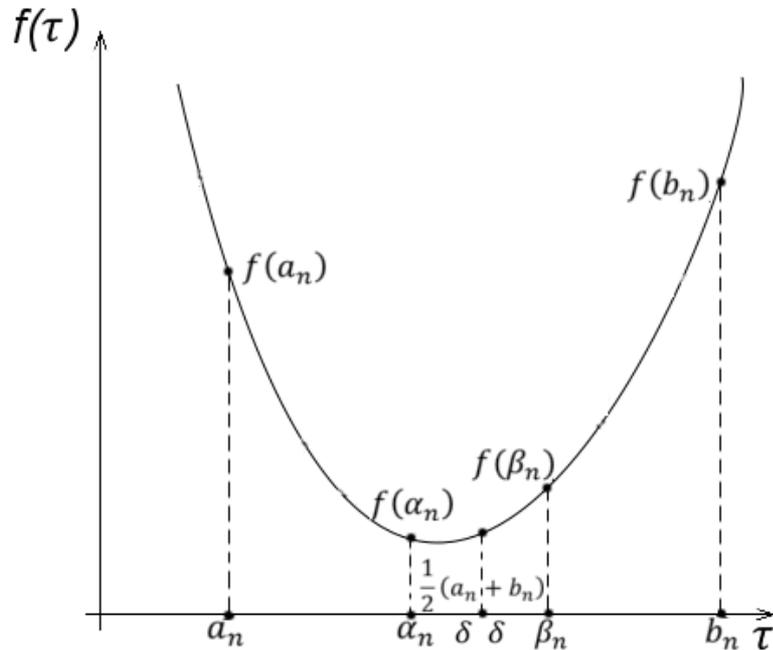
$$f(\alpha_n) > f(\beta_n)$$

$$a_{n+1} := \alpha_n$$

$$b_{n+1} := b_n$$



Dichotomous search method



$$\alpha_n = \frac{1}{2}(a_n + b_n) - \delta$$

$$\beta_n = \frac{1}{2}(a_n + b_n) + \delta$$

$$N = ? \text{ for } \varepsilon = 0.01, \Delta = b - a = 1$$

Input data: $a_0, b_0, \varepsilon, \delta$

Step 0: $n = 0$

Step 1: $\alpha_n = \frac{1}{2}(a_n + b_n) - \delta$

$$\beta_n = \frac{1}{2}(a_n + b_n) + \delta$$

Step 2: If $f(\alpha_n) \leq f(\beta_n)$ then
 $a_{n+1} := a_n, b_{n+1} := \beta_n$,
otherwise

$$a_{n+1} := \alpha_n, b_{n+1} := b_n.$$

Step 3: If $|b_{n+1} - a_{n+1}| \geq \varepsilon$ then

$n := n + 1$, go to 1,

otherwise

$$\tilde{\tau} = \frac{1}{2}(a_{n+1} + b_{n+1}) \text{ (STOP)}$$



Estimation of steps procedure number

- $\Delta_0 = b_0 - a_0 = 1$ initial length of interval
- $\Delta_1 = \frac{1}{2} \Delta_0$ interval length after one step
- $\Delta_2 = \frac{1}{2} \Delta_1 = \left(\frac{1}{2}\right)^2 \Delta_0$ interval length after two step
- \vdots
- $\Delta_N = \frac{1}{2} \Delta_{N-1} = \dots = \left(\frac{1}{2}\right)^N \Delta_0$ interval length after N-th step
- We expect, that after N steps interval length will be less then ε .
Then the number of steps must fulfil the following condition:

$$\Delta_N = \left(\frac{1}{2}\right)^N \Delta_0 \leq \varepsilon \quad \text{we divide by } \Delta_0$$



Estimation of steps procedure number

- $\left(\frac{1}{2}\right)^N \leq \frac{\varepsilon}{\Delta_0}$ log both sides
- $\ln\left(\frac{1}{2}\right)^N \leq \ln\frac{\varepsilon}{\Delta_0}$ consequently
- $N \ln\left(\frac{1}{2}\right) \leq \ln\frac{\varepsilon}{\Delta_0}$ dividing both sides by $\ln\left(\frac{1}{2}\right)$.

Because $\ln\left(\frac{1}{2}\right)$ is a negative number we change sign of inequality

Then the minimum number of procedure steps must fulfill the following condition:

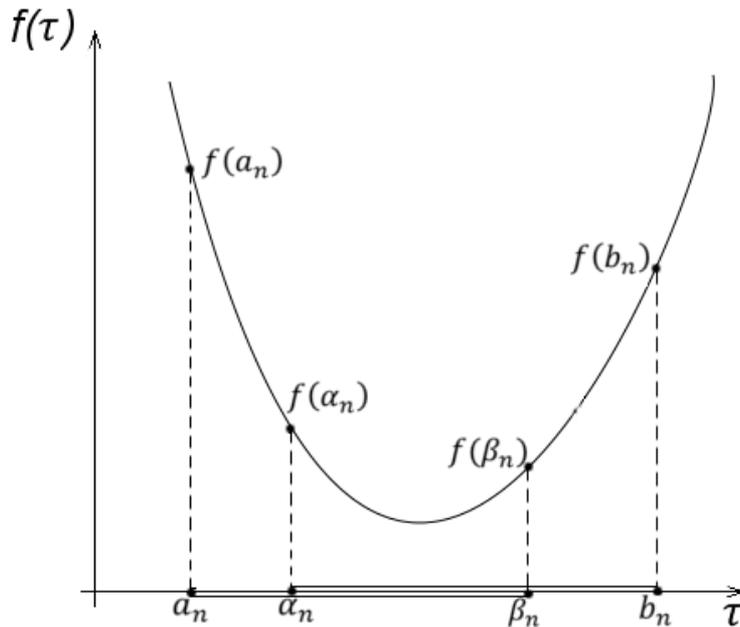
$$\bullet \quad N \geq \frac{\ln\frac{\varepsilon}{\Delta_0}}{\ln\frac{1}{2}} = \frac{\ln 0.01}{\ln 0.5} = \frac{4.605}{0.693} \approx 7$$

Because in each step we must calculate two Times value of function then necessary number of function calculation is:

- Number of function calculation is: $= 2N = 14$



The γ section method



$$\frac{b_n - \alpha_n}{b_n - a_n} = \frac{\beta_n - a_n}{b_n - a_n} = \gamma$$

$$\alpha_n = b_n + \gamma(a_n - b_n)$$

$$\beta_n = a_n + \gamma(b_n - a_n)$$

$N = ?$ for $\varepsilon = 0.01, \Delta = b - a = 1$

Input data: $a_0, b_0, \varepsilon, \gamma$

Step 0: $n = 0$

Step 1: $\alpha_n = b_n + \gamma(a_n - b_n)$
 $\beta_n = a_n + \gamma(b_n - a_n)$

Step 2: If $f(\alpha_n) \leq f(\beta_n)$ then
 $a_{n+1} := a_n, b_{n+1} := \beta_n$,
 otherwise

$a_{n+1} := \alpha_n, b_{n+1} := b_n$.

Step 3: If $|b_{n+1} - a_{n+1}| \geq \varepsilon$ then
 $n := n + 1$, go to 1,
 otherwise

$\tilde{\tau} = \frac{1}{2}(a_{n+1} + b_{n+1})$ (STOP)



Estimation of steps procedure number

- $\Delta_0 = b_0 - a_0 = 1$ initial length of interval , $\varepsilon=0.01$
- $\Delta_1 = \gamma\Delta_0$ interval length after one step
- $\Delta_2 = \gamma\Delta_1 = (\gamma)^2\Delta_0$ interval length after two step
- \vdots
- $\Delta_N = \gamma\Delta_{N-1} = \dots = (\gamma)^N\Delta_0$ interval length after N-th step
- We expect, that after N steps interval length will be less then ε .

Then the number of steps must fulfil the following condition:

$$\Delta_N = (\gamma)^N \Delta_0 \leq \varepsilon \quad \text{we divide by } \Delta_0$$



Estimation of steps procedure number

- $(\gamma)^N \leq \frac{\varepsilon}{\Delta_0}$ log both sides
- $\ln(\gamma)^N \leq \ln \frac{\varepsilon}{\Delta_0}$ consequently
- $N \ln(\gamma) \leq \ln \frac{\varepsilon}{\Delta_0}$ dividing both sides by $\ln(\gamma)$.

Because $\ln(\gamma)$ is a negative number we change sign of nonequality

Then the minimum number of procedure steps must fulfill the following condition:

- $$N \geq \frac{\ln \frac{\varepsilon}{\Delta_0}}{\ln \gamma}$$

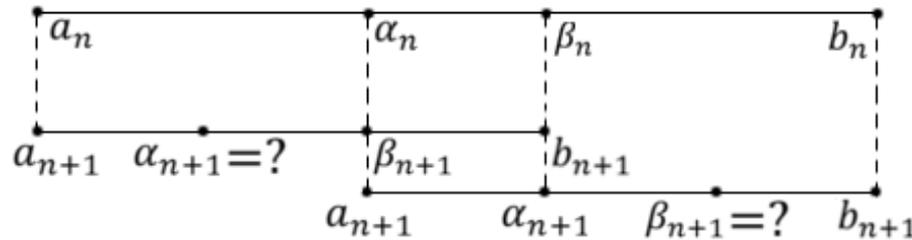
Because in each step we must calculate two Times value of function then necessary number of function calculation is:

Number of function calculation is = $2N$

$$\gamma = ??$$



The golden section method



$$\gamma^2 + \gamma - 1 = 0$$

$$\gamma = \frac{\sqrt{5} - 1}{2} \approx 0.618$$

1. $\frac{\beta_{n+1} - a_{n+1}}{b_{n+1} - a_{n+1}} = \gamma$, which gives

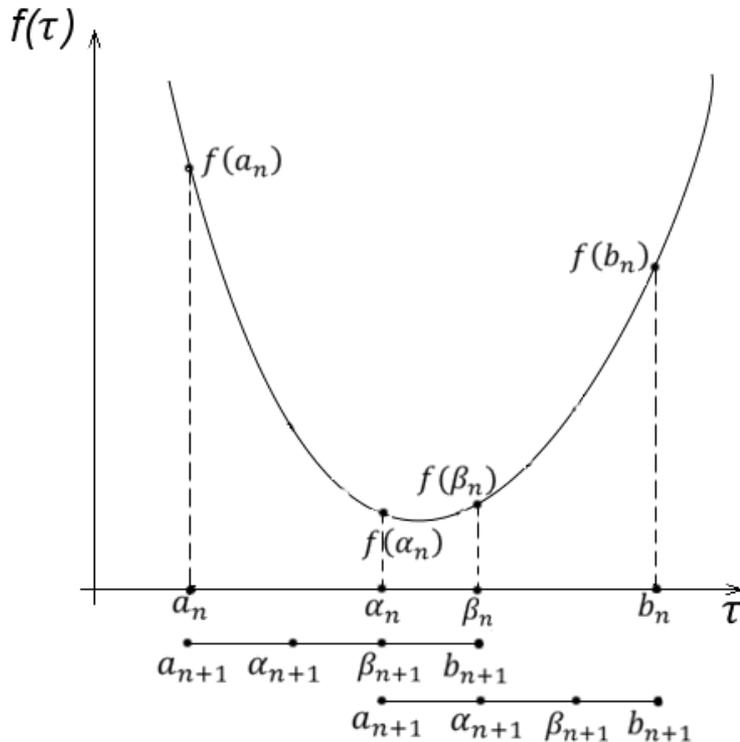
$$\frac{\alpha_n - a_n}{\beta_n - a_n} = \frac{b_n + \gamma(a_n - b_n) - a_n}{a_n + \gamma(b_n - a_n) - a_n} = \frac{b_n - a_n + \gamma(a_n - b_n)}{\gamma(b_n - a_n)} = \frac{1}{\gamma} - 1 = \gamma$$

2. $\frac{b_{n+1} - \alpha_{n+1}}{b_{n+1} - a_{n+1}} = \gamma$, which gives

$$\frac{b_n - \beta_n}{b_n - a_n} = \frac{b_n - a_n + \gamma(b_n - a_n)}{b_n - b_n - \gamma(b_n - a_n)} = \frac{b_n - a_n + \gamma(b_n - a_n)}{\gamma(b_n - a_n)} = \frac{1}{\gamma} - 1 = \gamma$$



The golden section method



Input data: $a_0, b_0, \varepsilon, \gamma = \frac{\sqrt{5}-1}{2}$

Step 0: $n = 0$

$$\alpha_0 = b_0 + \gamma(a_0 - b_0)$$

$$\beta_0 = a_0 + \gamma(b_0 - a_0)$$

Step 1: If $|b_n - a_n| < \varepsilon$, then

$$\tilde{\tau} = \frac{1}{2}(a_n + b_n) \text{ (STOP)}$$

otherwise go to 2

Step 2: If $f(\alpha_n) \leq f(\beta_n)$ then

$$a_{n+1} := a_n, b_{n+1} := \beta_n,$$

$$\beta_{n+1} := \alpha_n, \alpha_{n+1} := \beta_n + \gamma(a_n - b_n)$$

$$n := n + 1, \text{ go to 1}$$

otherwise

$$a_{n+1} := \alpha_n, b_{n+1} := b_n,$$

$$\alpha_{n+1} := \beta_n, \beta_{n+1} := \alpha_n + \gamma(b_n - a_n)$$

$$n := n + 1, \text{ go to 1}$$

$$\gamma^2 + \gamma - 1 = 0$$

$$\gamma = \frac{\sqrt{5}-1}{2} \approx 0.618$$

$$N = ? \text{ for } \varepsilon = 0.01, \Delta = b - a = 1$$



Estimation of steps procedure number

- $(\gamma)^N \leq \frac{\varepsilon}{\Delta_0}$ log both sides
- $\ln(\gamma)^N \leq \ln \frac{\varepsilon}{\Delta_0}$ consequently
- $N \ln(\gamma) \leq \ln \frac{\varepsilon}{\Delta_0}$ dividing both sides by $\ln(\gamma)$.

Because $\ln(\gamma)$ is a negative number we change sign of nonequality

Then the minimum number of procedure steps must fulfill the following condition

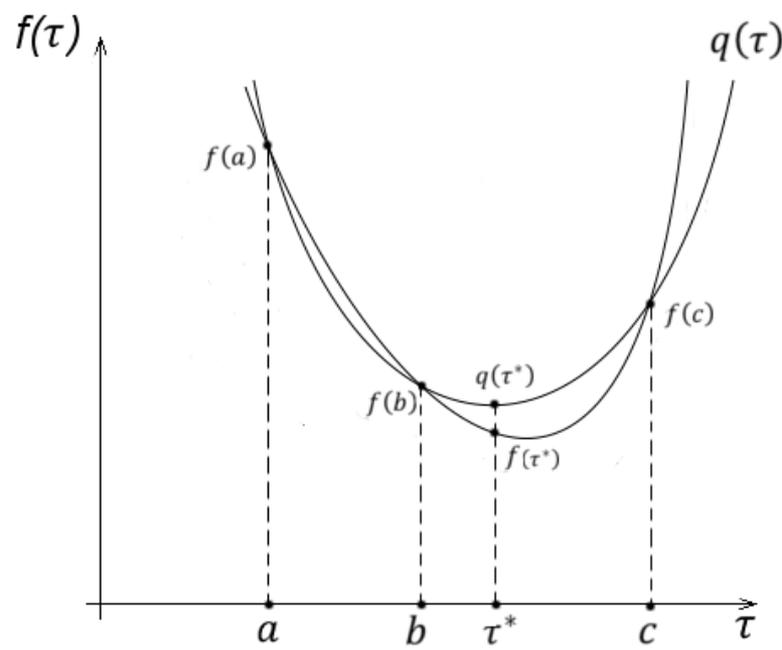
- $N \geq \frac{\ln \frac{\varepsilon}{\Delta_0}}{\ln \gamma}$ now $\gamma = \frac{\sqrt{5}-1}{2} \approx 0.618$

Because in each step we must calculate two Times value of function then necessary number of function calculation is:

- $\ln \gamma = \ln 0.618 = -0.481$
- $N \geq \frac{\ln \frac{\varepsilon}{\Delta_0}}{\ln \gamma} = \frac{\ln 0.01}{\ln 0.618} = \frac{4.605}{0.481} \approx 9,578 \approx 10$
- Number of function calculation is $N+1 = 10+1 = 11$



Quadratic-fit line search method



$$a < b < c$$

$$f(a) \geq f(b)$$

$$f(b) \leq f(c)$$

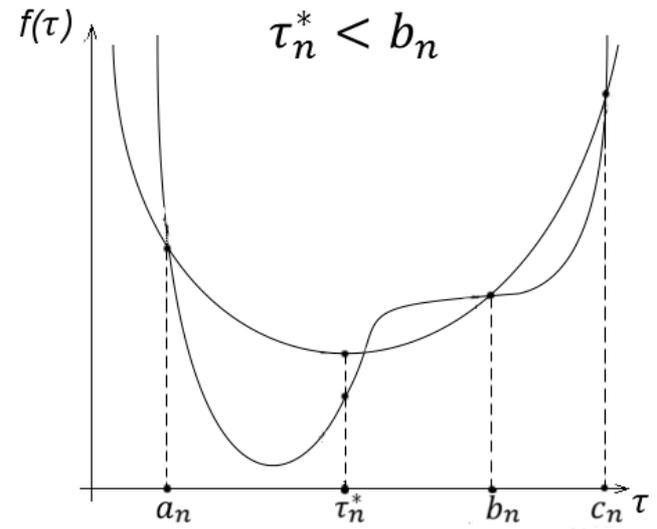
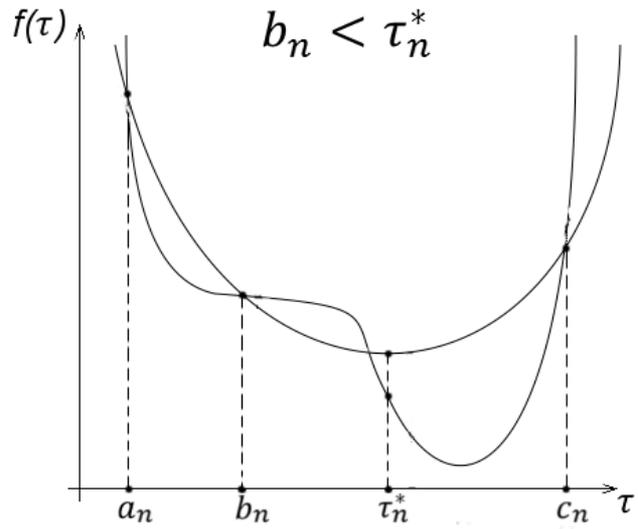
$q(\tau)$ – quadratic-fit function
 τ^* - minimum of the function $q(\tau)$

$$q(\tau) = \frac{f(a)(\tau - b)(\tau - c)}{(a - b)(a - c)} + \frac{f(b)(\tau - a)(\tau - c)}{(b - a)(b - c)} + \frac{f(c)(\tau - a)(\tau - b)}{(c - a)(b - c)}$$

$$\tau^* = \frac{1}{2} \frac{f(a)(b^2 - c^2) + f(b)(c^2 - a^2) + f(c)(a^2 - b^2)}{f(a)(b - c) + f(b)(c - a) + f(c)(a - b)}$$



Quadratic-fit line search method



$$f(b_n) \geq f(\tau_n^*)$$

$$a_{n+1} := b_n$$

$$b_{n+1} := \tau_n^*$$

$$c_{n+1} := c_n$$

$$f(b_n) < f(\tau_n^*)$$

$$a_{n+1} := a_n$$

$$b_{n+1} := b_n$$

$$c_{n+1} := \tau_n^*$$

$$f(b_n) \geq f(\tau_n^*)$$

$$a_{n+1} := a_n$$

$$b_{n+1} := \tau_n^*$$

$$c_{n+1} := b_n$$

$$f(b_n) \geq f(\tau_n^*)$$

$$a_{n+1} := \tau_n^*$$

$$b_{n+1} := b_n$$

$$c_{n+1} := c_n$$

$$|c_{n+1} - a_{n+1}| < \varepsilon \quad \tilde{\tau} = \tau_{n+1}^*$$



Quadratic-fit line search method

Input data: $a_0, b_0, c_0, \varepsilon$

Step 0: $n = 0$

$$\text{Step 1: } \tau_n = \frac{1}{2} \frac{f(a_n)(b_n^2 - c_n^2) + f(b_n)(c_n^2 - a_n^2) + f(c_n)(a_n^2 - b_n^2)}{f(a_n)(b_n - c_n) + f(b_n)(c_n - a_n) + f(c_n)(a_n - b_n)}$$

Step 2: If $b_n < \tau_n$ then go to 3

otherwise

If $f(b_n) \geq f(\tau_n)$ then $a_{n+1} := a_n, b_{n+1} := \tau_n, c_{n+1} := b_n$ go to 4

otherwise $a_{n+1} := \tau_n, b_{n+1} := b_n, c_{n+1} := c_n$ go to 4

Step 3: If $f(b_n) \geq f(\tau_n)$ to $a_{n+1} := b_n, b_{n+1} := \tau_n, c_{n+1} := c_n$ go to 4

otherwise $a_{n+1} := a_n, b_{n+1} := b_n, c_{n+1} := \tau_n$ go to 4

Step 4: If $|c_{n+1} - a_{n+1}| \geq \varepsilon$ then $n := n + 1$ go to 1

otherwise $\tilde{\tau} = \frac{1}{2}(a_{n+1} + c_{n+1})$ (STOP)

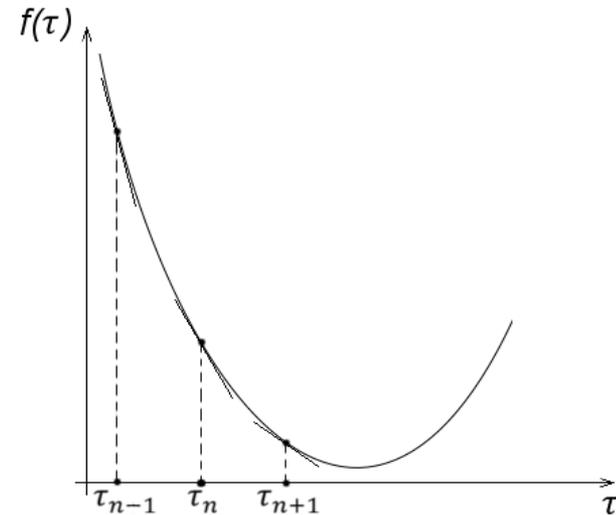


Line search using derivatives

$$\tau_{n+1} = \tau_n - \gamma_n f'(\tau_n) \quad \gamma_n > 0, \tau_0$$

$$\lim_{n \rightarrow \infty} \gamma_n = \gamma \quad \sum_{n=0}^{\infty} \gamma_n = \infty$$

e.g. $|\tau_{n+1} - \tau_n| < \varepsilon$ (STOP)



$$\begin{aligned} \tau_0 \\ \tau_1 &= \tau_0 - \gamma_0 f'(\tau_0) \\ \tau_2 &= \tau_1 - \gamma_1 f'(\tau_1) = \tau_0 - \gamma_0 f'(\tau_0) - \gamma_1 f'(\tau_1) \\ \tau_{n+1} &= \tau_n + \gamma_n f'(\tau_n) = \dots = \tau_0 - \gamma_0 f'(\tau_0) - \gamma_1 f'(\tau_1) - \dots - \gamma_n f'(\tau_n) \end{aligned}$$

$$|\tau_{n+1} - \tau_0| = \left| \sum_{k=0}^n \gamma_k f'(\tau_k) \right| \leq \sum_{k=0}^n \gamma_k |f'(\tau_k)| \leq \max_{0 < k < n} |f'(\tau_k)| \sum_{k=0}^n \gamma_k$$

$$|\tau_{\infty} - \tau_0| \leq \sum_{k=0}^{\infty} \gamma_k = \infty$$



Line search using sign of derivatives

$$\tau_{n+1} = \tau_n - \vartheta_n \text{sign}[f'(\tau_n)]$$

$$\gamma_n f'(\tau_n) = \gamma_n |f'(\tau_n)| * \text{sign} f'(\tau_n) = \vartheta_n \text{sign}[f'(\tau_n)], \text{ where } \vartheta_n = \gamma_n |f'(\tau_n)|$$

$$\vartheta_n > 0$$

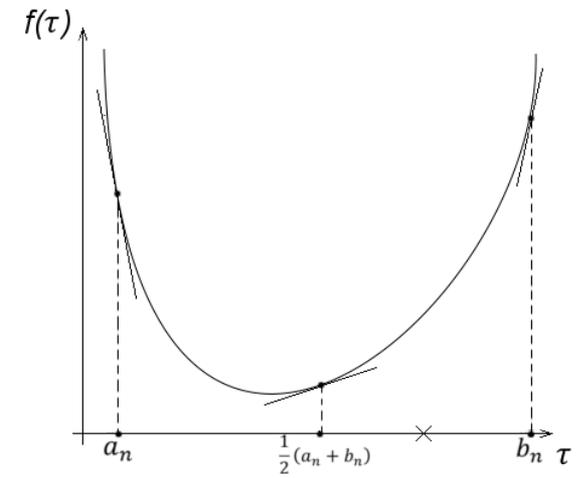
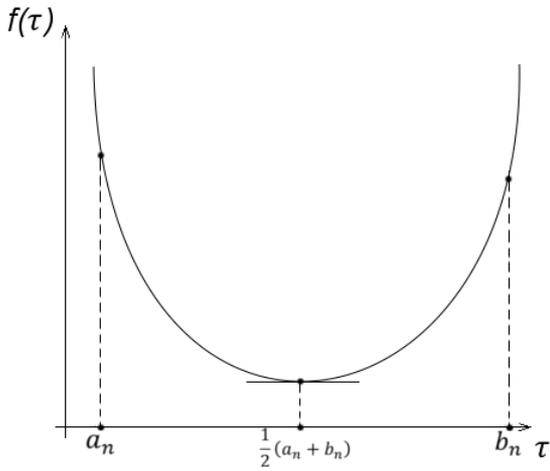
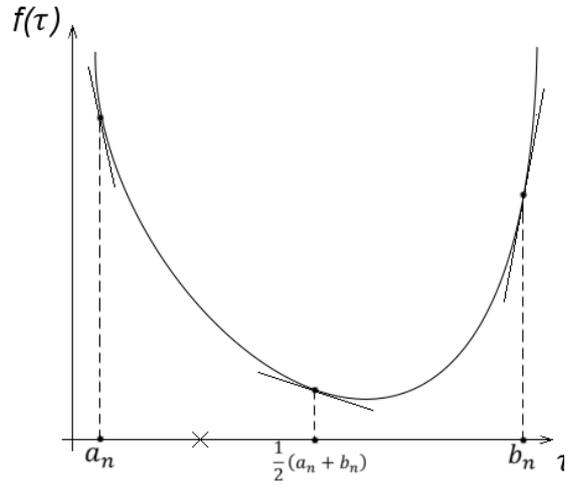
$$\lim_{n \rightarrow \infty} \vartheta_n = 0, \text{ because } \lim_{n \rightarrow \infty} |f'(\tau_n)| = 0, \lim_{n \rightarrow \infty} \gamma_n = \gamma$$

$$\sum_{n=0}^{\infty} \vartheta_n = \infty \quad \lim_{n \rightarrow \infty} \vartheta_n = \lim_{n \rightarrow \infty} \gamma_n |f'(\tau_n)| = 0$$



Bolzano method

$sign a_n \neq sign b_n$



$$sign f'(a_n) = sign f'(\frac{1}{2}(a_n + b_n)) \quad f'(\frac{1}{2}(a_n + b_n)) = 0$$

$$a_{n+1} := \frac{1}{2}(a_n + b_n)$$

$$b_{n+1} := b_n$$

$$\tilde{\tau} := \frac{1}{2}(a_n + b_n)$$

$$sign f'(b_n)$$

$$= sign f'(\frac{1}{2}(a_n + b_n))$$

$$a_{n+1} := a_n$$

$$b_{n+1} := \frac{1}{2}(a_n + b_n)$$



Thank you for attention

