

# Computer Science

## Jerzy Świątek

# Systems Modelling and Analysis

*Choose yourself and new technologies*

## L.8 Integer programming



**HUMAN CAPITAL**  
HUMAN – BEST INVESTMENT!



Wrocław University of Technology

EUROPEAN  
SOCIAL FUND



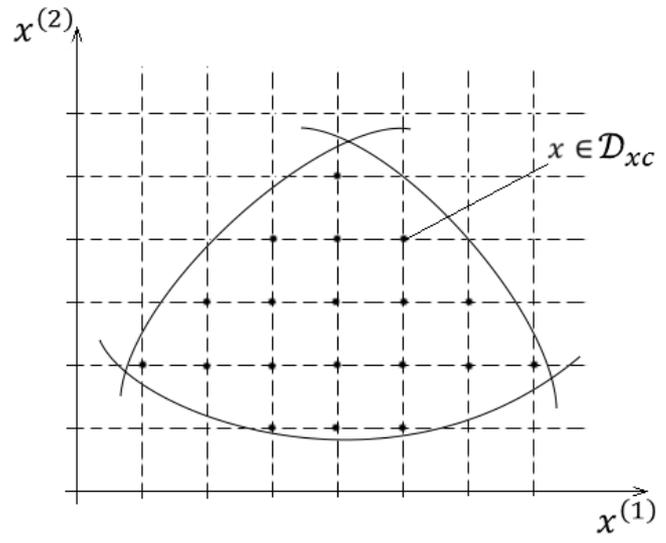
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# Integer programming – branch and bound method

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_{xc}} F(x)$$

$\mathcal{D}_{xc} = \mathcal{D}_x \cap \{x^{(s)} \in \mathcal{C} \mid s = 1, 2, \dots, S\}$  integer decision variables



Special case

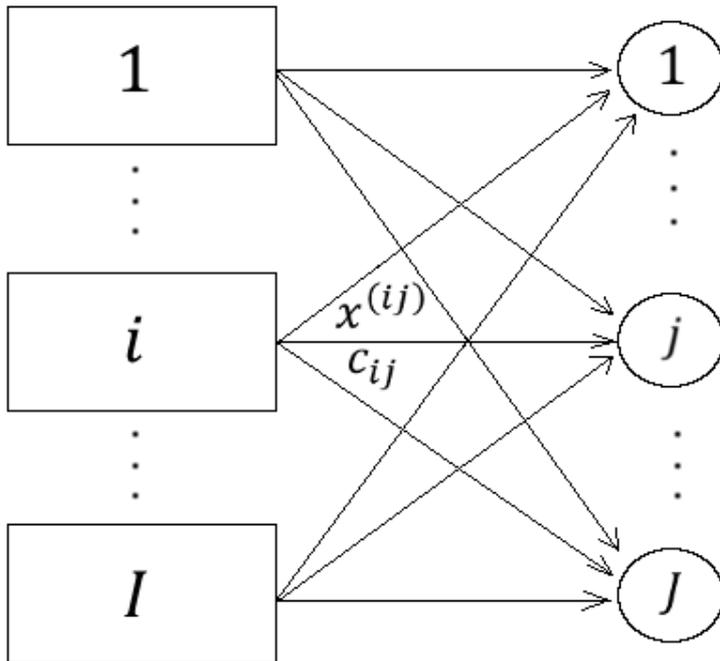
$\mathcal{D}_{xc} = \{x_1, x_2, \dots, x_k\}$  – finite set,  $k$  – large number

$\mathcal{D}_{xc} = \{0, 1\}$  - binary programming



# Classic problems

## Transportation problem



$I$  – the number of suppliers

$J$  – the number of recipients

$x^{(ij)}$  – the number of units of goods delivered  
by  $i$  – th supplier to  $j$  – th receiver

$c_{ij}$  – the cost of transport from  $i$  – th supplier to  
 $j$  – th receiver

$a_i$  – amount of goods belonging to  $i$  – th supplier

$b_j$  – demand for goods of  $j$  – th receiver

$i = 1, 2, \dots, I$        $j = 1, 2, \dots, J$



The goal function – the total cost of transport:

$$F(x) = \sum_{i=1}^I \sum_{j=1}^J c_{ij} x^{(ij)}$$

Constraints:

$$\sum_{j=1}^J x^{(ij)} \leq a_i \quad i = 1, 2, \dots, I$$

$$\sum_{i=1}^I x^{(ij)} = b_j \quad j = 1, 2, \dots, J$$

$$x^{(ij)} \geq 0 \quad x^{(ij)} \in \mathcal{C}^+ \quad i = 1, 2, \dots, I \\ (w) \quad j = 1, 2, \dots, J$$



# The choice of variant

The set of  $I$  facilities is given and  $J$  methods of modernization are available. The choice of  $j$ -th method to modernize  $i$ -th facility involves costs  $c_{ij}$  ( $i = 1, 2, \dots, I$ ,  $j = 1, 2, \dots, J$ ). Revenues during modernization are  $d_{ij}$ . The problem is to assign methods of modernization to facilities in such a way, that total revenue do not fall below  $d$ , and the total cost of modernization is minimal.

Decision variables

$$x^{(ij)} = \begin{cases} 1 & i\text{-th facility is modernized using } j\text{-th method} \\ 0 & \text{otherwise} \end{cases}$$



The goal function

$$F(x) = \sum_{i=1}^I \sum_{j=1}^J c_{ij} x^{(ij)}$$

Constraints

$$\sum_{i=1}^I \sum_{j=1}^J d_{ij} x^{(ij)} \geq d$$

$$\sum_{i=1}^I x^{(ij)} = 1 \quad i = 1, 2, \dots, I$$

only one method may  
be chosen to  
modernize  $i$ -th facility

$$x^{(ij)} \in \{0,1\} \quad i = 1, 2, \dots, I, j = 1, 2, \dots, J$$



# Knapsack problem

Given a set of items, each with a mass and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.

Notation:

$c_s$  – value of  $s$ -th item

$d_s$  – mass of  $s$ -th item

$d$  – permissible mass

The goal function:

$$F(x) = \sum_{s=1}^S c_s x^{(s)}, x^{(s)} = \begin{cases} 1 & s\text{-th item} \\ 0 & \text{otherwise} \end{cases}$$

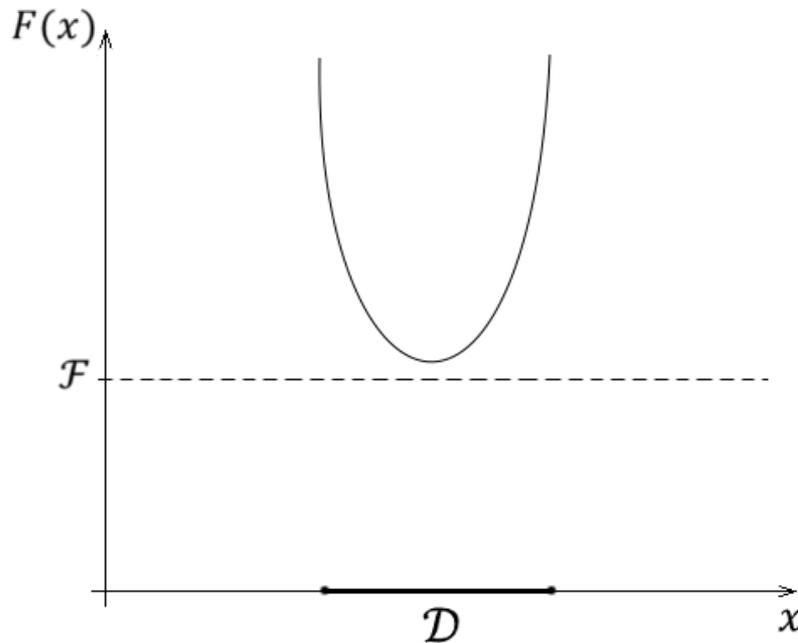
Constraints:

$$\sum_{s=1}^S d_s x^{(s)} \leq d$$



# Branch and bound method

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_{xc}} F(x)$$



$\mathcal{F}(\mathcal{D})$  – estimation of the function  $F$  on the set  $\mathcal{D}$

$$\mathcal{F}(\mathcal{D}) = \inf_{x \in \mathcal{D}} F(x)$$

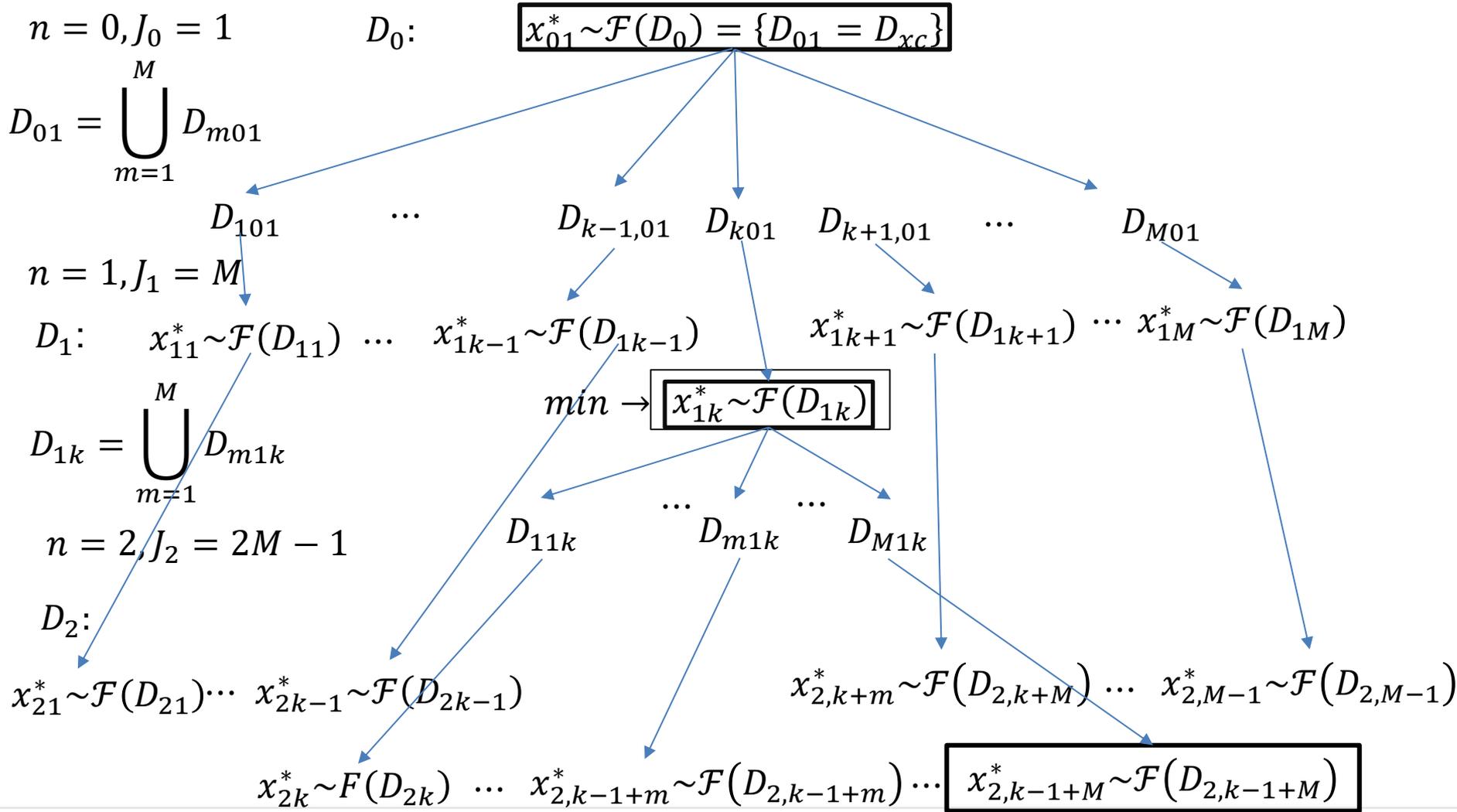
Properties:

$$\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}$$

$$\mathcal{F}(\mathcal{D}_1) \geq \mathcal{F}(\mathcal{D}_2)$$

$$\mathcal{F}(\{x\}) = F(x)$$

$$\mathcal{F}(\emptyset) = \infty$$





Step 0:  $\mathcal{D}_0 = \{\mathcal{D}_{xc} = \mathcal{D}_{01}\}, n = 0, J_0 = 1$

Step 1: Determine a set  $\mathcal{D}^* \in \mathcal{D}_n$

$$\mathcal{F}(\mathcal{D}^*) = \min_{\mathcal{D} \in \mathcal{D}_n} \mathcal{F}(\mathcal{D})$$

Step 2: Checking whether  $\mathcal{D}^*$  is a set? ( $\{x^*\} = \mathcal{D}^*$ )

or  $x^* \sim \mathcal{F}(\mathcal{D}^*)$  i.e.  $\mathcal{F}(\mathcal{D}^*) = F(x^*)$   $x^* \in \mathcal{D}^*$  (?) then  $x^*$  optimal solution

STOP

Step 3:  $\mathcal{D}^* = \mathcal{D}_{nk}$  is split up into  $M$  disjoint sets

$$\mathcal{D}_{1nk} \mathcal{D}_{2nk} \dots \mathcal{D}_{Mnk} \quad \mathcal{D}_{nk} = \bigcup_{m=1}^M \mathcal{D}_{mnk}$$

Step 4:  $\mathcal{D}^* = \mathcal{D}_{nk}$

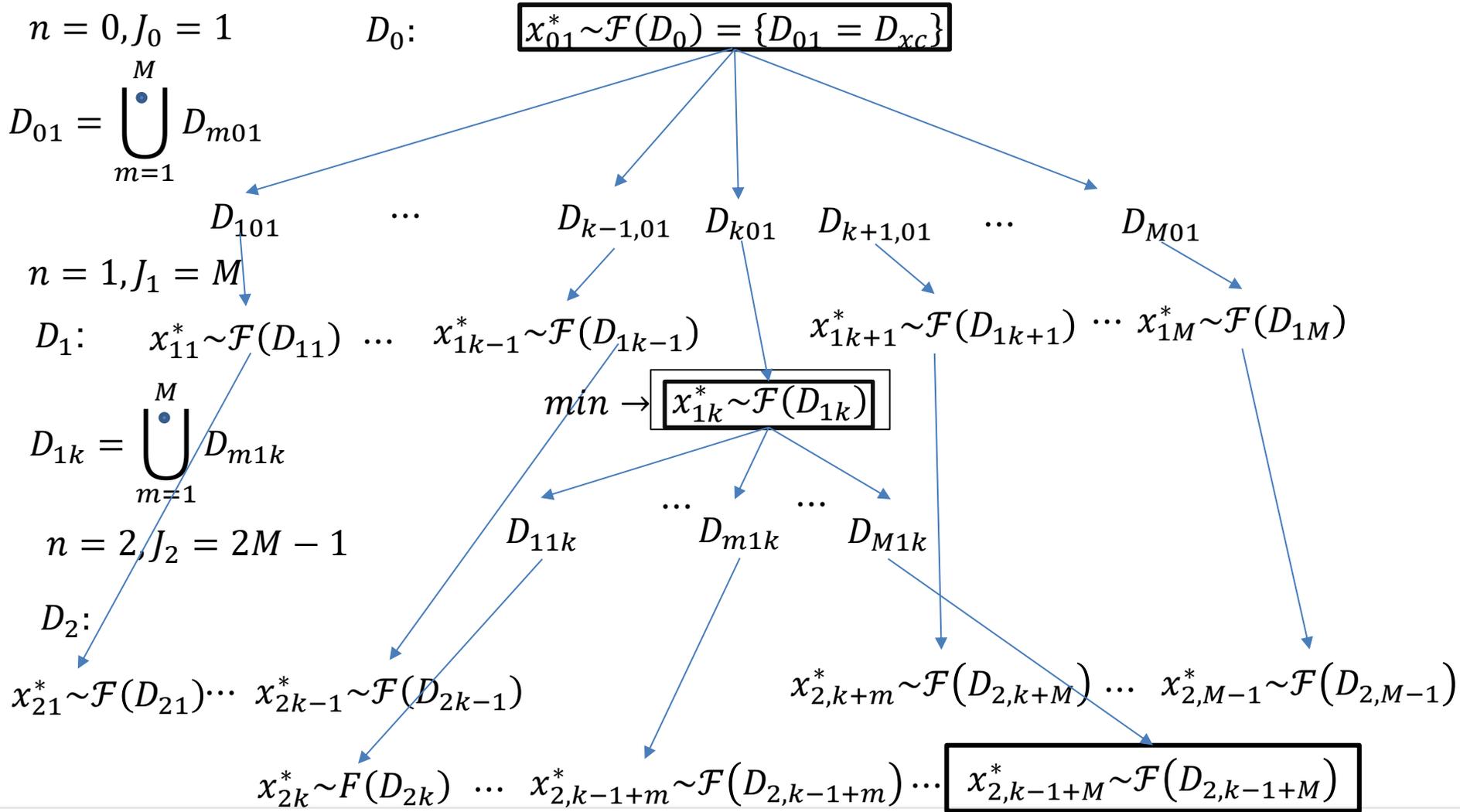
$$\mathcal{D}_{n+1} = \mathcal{D}_n \cup \{\mathcal{D}_{1nk}, \mathcal{D}_{2nk}, \dots, \mathcal{D}_{Mnk}\} \setminus \mathcal{D}_{nk}$$

$$\mathcal{D}_{n+1,j} = \mathcal{D}_{nj} \quad j = 1, 2, \dots, k - 1$$

$$\mathcal{D}_{n+1,j} = \mathcal{D}_{mnk} \quad j = k + m, m = 1, 2, \dots, M$$

$$\mathcal{D}_{n+1,j} = \mathcal{D}_{ni} \quad j = k + M + i, i = k + 1, \dots, J_n, J_{n+1} = J_n + M - 1$$

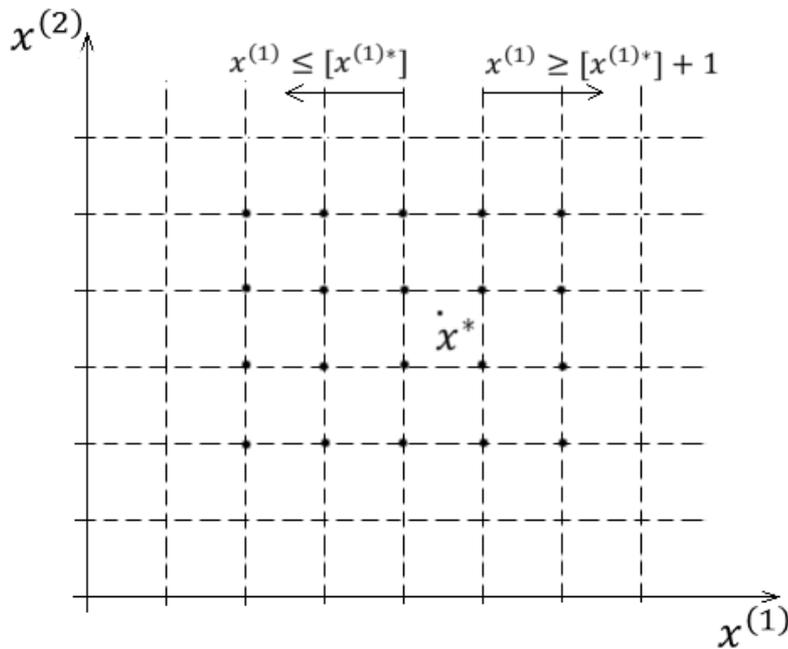
Go to step 1.





$$\mathcal{F}(\mathcal{D}_{xc}) = \min_{x \in \mathcal{D}_x} F(x)$$

Division of the set of feasible solutions



$$x^{(s)} \leq [x^{(s)*}]$$

$$x^{(s)} \geq [x^{(s)*}] + 1$$

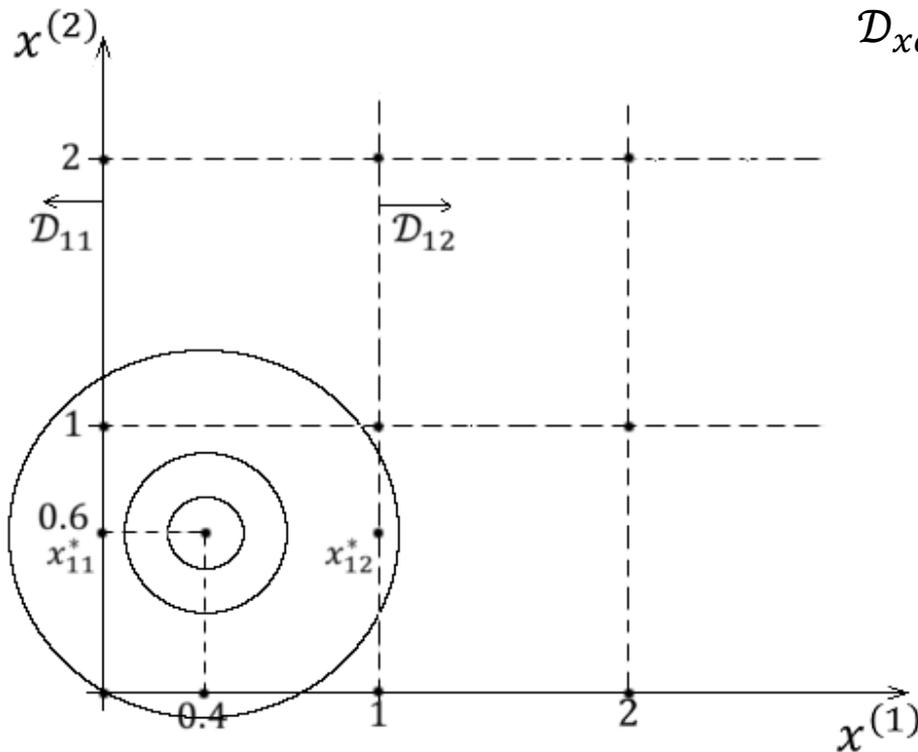
$$\mathcal{D}_{11} = \mathcal{D}_{01} \cap \{x^{(1)} \leq [x^{(1)*}]\}$$

$$\mathcal{D}_{12} = \mathcal{D}_{01} \cap \{x^{(1)} \geq [x^{(1)*}] + 1\}$$



$$F(x) = (x^{(1)} - 0.4)^2 + (x^{(2)} - 0.6)^2$$

$$\mathcal{D}_{xc} = \{x \in R^2, x^{(1)}, x^{(2)} \in \mathcal{C}\}$$





1.  $\mathcal{F}(\mathcal{D}_{xc}) = \min_{R^2} F(x)$

$$x_{01}^* = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{01}) = 0$$

2.  $\mathcal{D}_{11} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \wedge x^{(1)} \leq [0.4] = 0\}$

$$\mathcal{D}_{12} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \wedge x^{(1)} \geq [0.4] + 1 = 1\}$$

$$\mathcal{F}(\mathcal{D}_{11}) = \min_{x^{(1)} \leq 0} F(x)$$

$$x_{11}^* = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{11}) = (0.4)^2 + 0^2 = 0.16$$

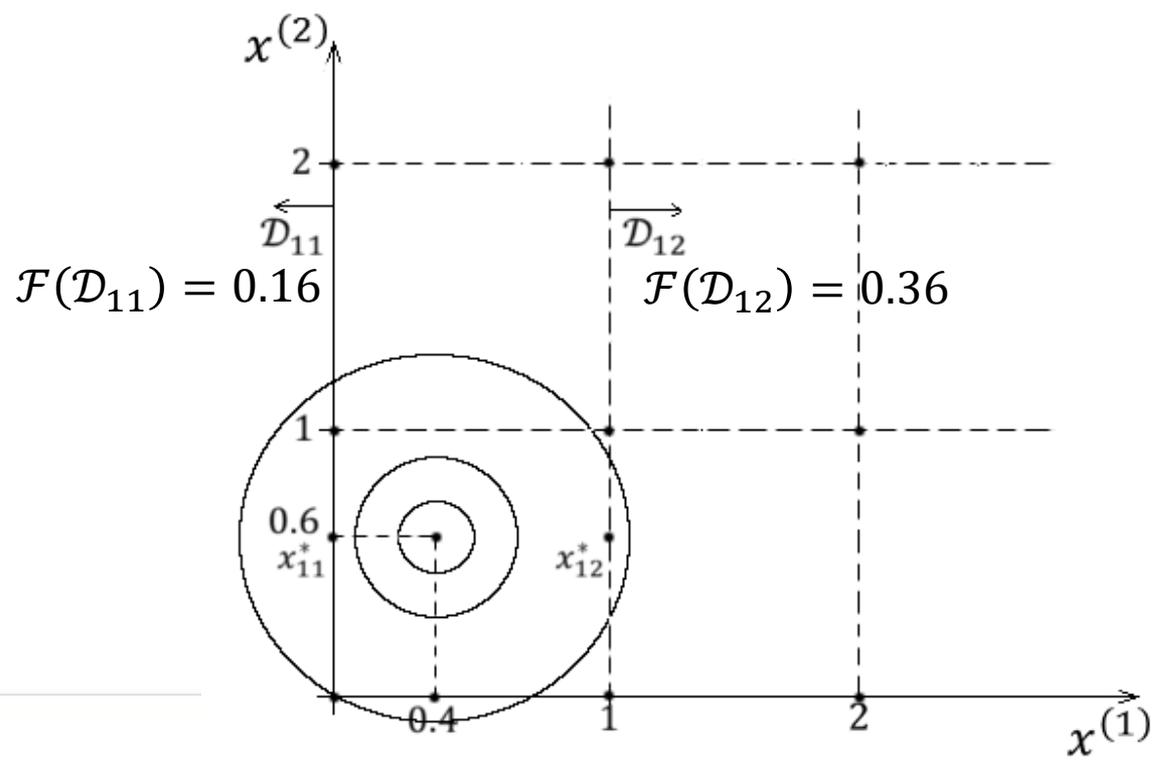
$$\mathcal{F}(\mathcal{D}_{12}) = \min_{x^{(1)} \geq 1} F(x)$$

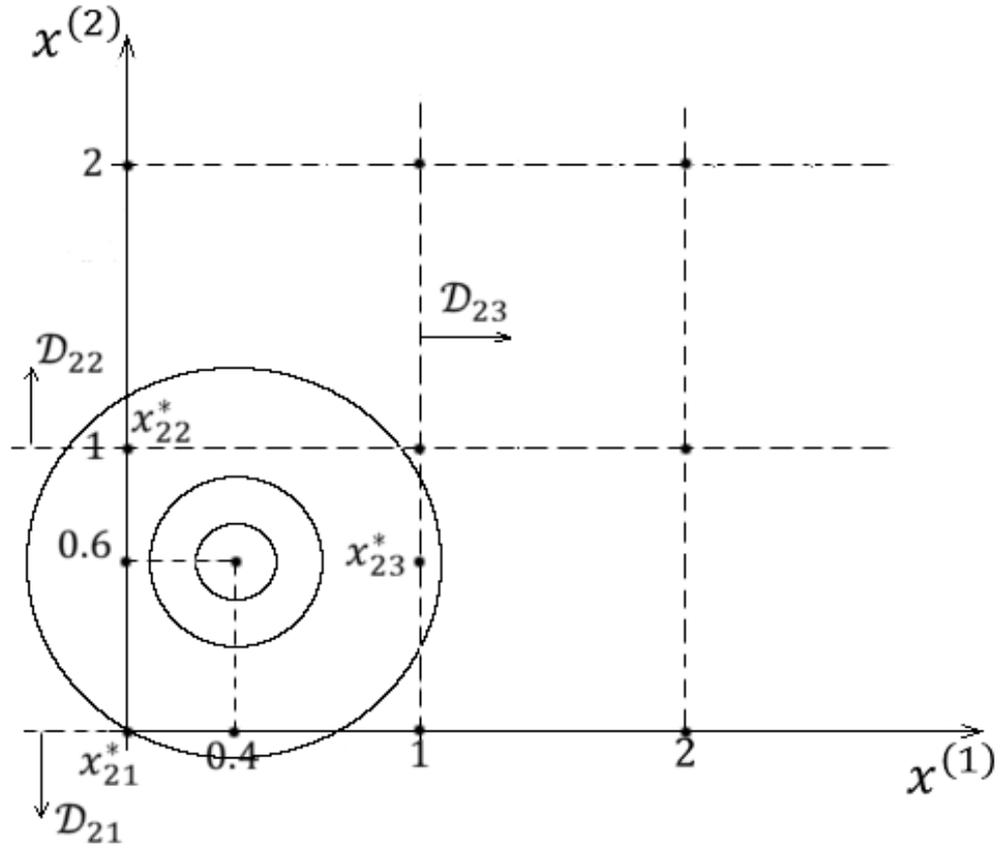
$$x_{12}^* = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{12}) = (0.6)^2 + 0^2 = 0.36$$



$$F(x) = (x^{(1)} - 0.4)^2 + (x^{(2)} - 0.6)^2$$

$$\mathcal{D}_{xc} = \{x \in R^2, x^{(1)}, x^{(2)} \in \mathcal{C}\}$$







3.  $n = 2$

$$\mathcal{D}_{21} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \wedge x^{(1)} \leq 0, x^{(2)} \leq [0.6] = 0\}$$

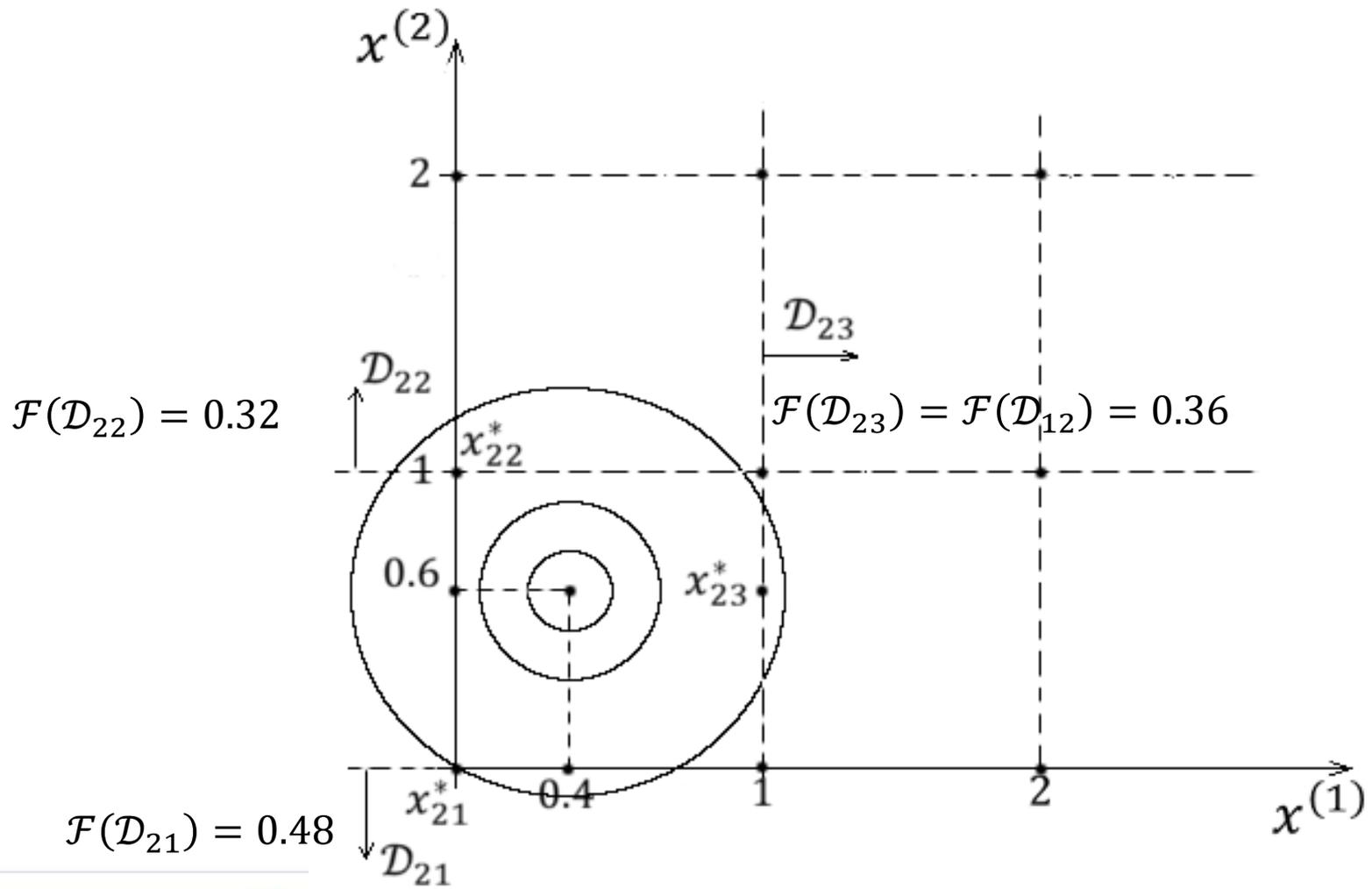
$$\mathcal{F}(\mathcal{D}_{21}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \leq 0}} F(x) \quad x_{21}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{21}) = 0.16 + 0.32 = 0.48$$

$$\mathcal{D}_{22} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \quad x^{(1)} \leq 0 \quad x^{(2)} \geq [0.6] + 1 = 1\}$$

$$\mathcal{F}(\mathcal{D}_{22}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \geq 1}} F(x) \quad x_{22}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{22}) = \mathbf{0.16} + \mathbf{0.16} = \mathbf{0.32}$$

$$\mathcal{D}_{23} = \mathcal{D}_{12}$$

$$\mathcal{F}(\mathcal{D}_{23}) = 0.36$$





$$n = 0, J_0 = 1 \quad D_0: \quad x_{01}^* = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(D_{01}) = \min_{R^2} F(x) = 0$$

$$x^{(1)} \leq [0.4] = 0$$

$$x^{(1)} \geq [0.4] + 1 = 1$$

$$n = 1, J_1 = 2$$

$$D_1: \quad x_{11}^* = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(D_{11}) = \min_{x^{(1)} \leq 0} F(x) = 0.16$$

$$x_{12}^* = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(D_{12}) = \min_{x^{(1)} \geq 1} F(x) = 0.36$$

$$x^{(1)} \leq 0, x^{(2)} \leq [0.6] = 0$$

$$x^{(2)} \geq [0.6] + 1 = 1$$

$$n = 2, J_2 = 3$$

$$D_2: \quad x_{21}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sim \mathcal{F}(D_{21}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \leq 0}} F(x) = 0.48$$

$$x_{22}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sim \mathcal{F}(D_{22}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \geq 1}} F(x) = 0.32$$

$$D_{23} = D_{12}$$

$$x_{12}^* = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(D_{23}) = 0.36$$



# Thank you for attention

