# Computer Science

Jerzy Świątek

# Systems Modelling and Analysis

Choose yourself and new technologies

## L.9 Decision making under uncertainty

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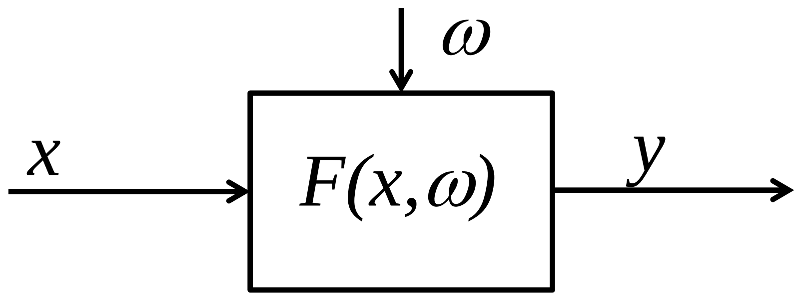
# UNCERTAINTY – RANDOM VARIABLE

# Decision making under uncertainty




$$x^* \to F\left(x^*, \omega\right) = \min_{x \in D_x(\omega)} F\left(x, \omega\right) \quad ???$$

# The choice of variant

Let us consider an example:

The network of $I$ mines to be modernized is given. There are $J$ variants of modernization, each involving costs $c_{ij}$ if $i$-th mine is modernized by $j$-th variant ($i = 1, 2, \dots, I, \quad j = 1, 2, \dots, J$). Each mine's spoil is $u_{ij}$ - if $i$-th mine is modernized by $j$-th variant. Contamination of each mine's spoil is $z_{ij}$ - if $i$-th mine is modernized by $j$-th variant. The task is to assign variants to mines in such a way that total spoil do not fall below *u*, contaminations of spoil meet the requirements of the market (defined by the upper limit *z*) and total costs are as small as possible.

# Decision making under uncertainty

$x_{ij}$ - decision variable

$$x_{ij} = \begin{cases} 1 & \text{- } i\text{–th mine modernized by } j\text{–th variant} \\ 0 & \text{- otherwise} \end{cases}$$

$$F(x,c) = \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x_{ij}$$

$$\sum_{i=1}^{I} \sum_{j=1}^{J} u_{ij} x_{ij} \geq u$$

$$\sum_{j=1}^{J} z_{ij} x_{ij} \leq z \quad i = 1, 2, \ldots, I$$

$$\sum_{j=1}^{J} x_{ij} = 1 \quad i = 1, 2, \ldots, I$$

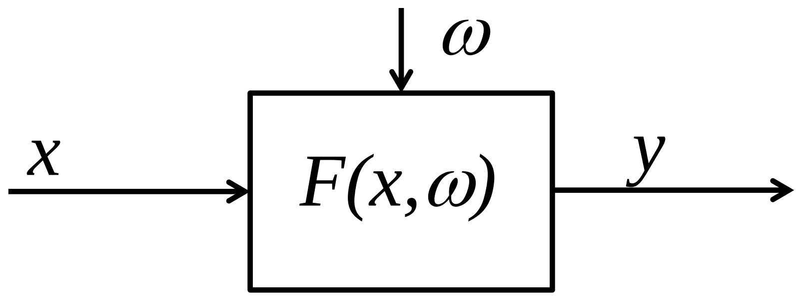
# Decision making under uncertainty

$\left\{\underline{c}_{ij}, \underline{u}_{ij}, \underline{z}_{ij}\ i=1,\dots,I,\ j=1,\dots,J\right\} = \underline{\omega}$ - uncertain (random) variables

$$F(x,\underline{c}) = \sum_{i=1}^{I}\sum_{j=1}^{J}\underline{c}_{ij}x_{ij} \qquad \Rightarrow F(x,\underline{\omega})$$

$$\left.\begin{aligned} &\sum_{i=1}^{I}\sum_{j=1}^{J}\underline{u}_{ij}x_{ij} \geq u \\ &\sum_{j=1}^{J}\underline{z}_{ij}x_{ij} \leq z \quad i=1,\dots,I \\ &\sum_{j=1}^{J}x_{ij} = 1 \quad i=1,2,\dots,I \end{aligned}\right\} \Rightarrow \mathscr{D}_x(\underline{\omega}) = \left\{x \in \mathscr{R}^S;\ \varphi_l(x,\underline{\omega}) = 0,\ l=1,\dots,L,\ \psi_m(x,\underline{\omega}) \leq 0,\ m=1,\dots,M\right\}$$

# Decision making under uncertainty




$$x^* \rightarrow F\left(x^*, \omega\right) = \min_{x \in D_x(\omega)} F\left(x, \omega\right) \quad ???$$

# Decision making under uncertainty

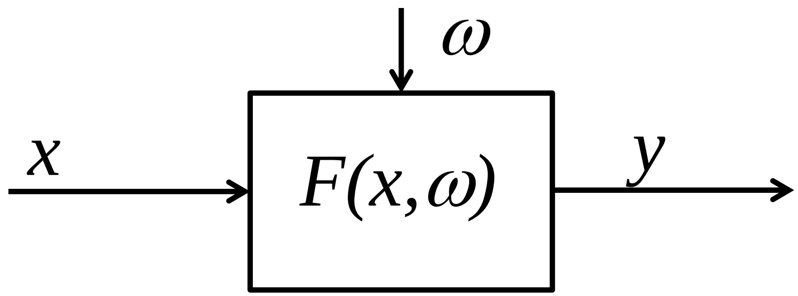
$$E_{\underline{c}}\left[F\left(x,\underline{c}\right)\right]=E_{\underline{c}}\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\underline{c}_{ij}x_{ij}\right]$$

$$E_{\underline{u}}\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\underline{u}_{ij}x_{ij}\right]\geq u$$

$$E_{\underline{z}}\left[\sum_{j=1}^{J}\underline{z}_{ij}x_{ij}\right]\leq z \quad i=1,2,\ldots,I$$

$$\sum_{j=1}^{J}x_{ij}=1 \quad i=1,2,\ldots,I$$

# Decision making under uncertainty




$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right]$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}\left[\mathscr{D}_x(\underline{\omega})\right] =$$

$$= \left\{ x \in \mathscr{R}^S ;\ \underset{\underline{\omega}}{E}\left[\varphi_l(x,\underline{\omega})\right] = 0,\ l = 1, \ldots, L,\ \underset{\underline{\omega}}{E}\left[\psi_m(x,\underline{\omega})\right] \le 0,\ m = 1, \ldots, M \right\}$$

$$x^* \rightarrow F(x^*) = \min_{x \in D_x} F(x)$$

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# Decision making under uncertainty

$\omega$ - value of random variable $\underline{\omega}$

$\omega \in \Omega$ - a continuous set,

$f_{\omega}(\omega)$ - probability density functions of random variable $\underline{\omega}$

Then:
$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right] = \int_{\Omega} F(x,\omega) f_{\omega}(\omega) d\omega$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}\left[\mathscr{D}_x(\underline{\omega})\right] = \Big\{ x \in \mathscr{R}^S ;\ \underset{\underline{\omega}}{E}\left[\varphi_l(x,\underline{\omega})\right] = \int_{\Omega} \varphi_l(x,\omega) f_{\omega}(\omega) d\omega = 0,\ l = 1, \dots, L,$$

$$\underset{\underline{\omega}}{E}\left[\psi_m(x,\underline{\omega})\right] = \int_{\Omega} \psi_m(x,\omega) f_{\omega}(\omega) d\omega \le 0,\ m = 1, \dots, M \Big\}$$

# Decision making under uncertainty

$\omega$ - value of random variable $\underline{\omega}$

$\omega \in \Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$ - a discrete set,

$P(\underline{\omega} = \omega_k) = p_k,\ k = 1, 2, \ldots, K$ - probability density functions of random variable $\underline{\omega}$

Then:

$$F(x) = \underset{\underline{\omega}}{E}[F(x, \underline{\omega})] = \sum_{k=1}^{K} F(x, \omega_k) p_k$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}[\mathscr{D}_x(\underline{\omega})] = \Big\{ x \in \mathscr{R}^S;\ \underset{\underline{\omega}}{E}[\varphi_l(x, \underline{\omega})] = \sum_{k=1}^{K} \varphi_l(x, \omega_k) p_k = 0,\ l = 1, \ldots, L,$$

$$\underset{\underline{\omega}}{E}[\psi_m(x, \underline{\omega})] = \sum_{k=1}^{K} \psi_m(x, \omega_k) p_k \le 0,\ m = 1, \ldots, M \Big\}$$

# Decision making under uncertainty

The problem of newspapers vendor.

A newspaper vendor makes an order for bundles of 40 newspapers. Quantity price of one newspaper is 0.80 euro, and vendors sells it for 1.10 euro. Demand for newspapers $\underline{\omega}$ is random variable. The „bad day" sale $\omega_1 = 50$ newspapers, the „moderate day" sale $\omega_2 = 100$ newspapers, and the „good day" $\omega_3 = 150$ newspapers. Probability of the „bad day" is $P(\underline{\omega} = \omega_1) = p_1 = 0.26$, of the „moderate day" is $P(\underline{\omega} = \omega_2) = p_2 = 0.40$, of the „good day" is $P(\underline{\omega} = \omega_3) = p_3 = 0.34$.

Decision variable *x* is the number of bundles the vendor should order.

*x* =1, 2, 3, 4, … ?

# Decision making under uncertainty

The goal function:

$$F(x,\underline{\omega}) = \begin{cases} 40x \times 0.30 & \text{if} \quad 40x < \underline{\omega} \\ \underline{\omega} \times 0.30 - (40x - \underline{\omega}) \times 0.80 & \text{if} \quad 40x \geq \underline{\omega} \end{cases}$$

Value of the goal function for x = 1, 2, 3 i 4

| ***x*** | $\omega_1 = 50$ | $\omega_2 = 100$ | $\omega_3 = 150$ |
|---|---|---|---|
| | $p_1 = 0.26$ | $p_2 = 0.40$ | $p_3 = 0.34$ |
| *x*=1 | 12 | 12 | 12 |
| *x*=2 | -9 | 24 | 24 |
| *x*=3 | -41 | 14 | 36 |
| *x*=4 | -73 | -18 | 37 |

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# Decision making under uncertainty

$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right] = \sum_{k=1}^{K} F(x,\omega_k)p_k$$

$$F(x=1) = \underset{\underline{\omega}}{E}\left[F(x=1,\underline{\omega})\right] = 12\times 0.26 + 12\times 0.4 + 12\times 0.34 = 12$$

$$F(x=2) = \underset{\underline{\omega}}{E}\left[F(x=2,\underline{\omega})\right] = -9\times 0.26 + 24\times 0.4 + 24\times 0.34 = 15.42$$

$$F(x=3) = \underset{\underline{\omega}}{E}\left[F(x=3,\underline{\omega})\right] = -41\times 0.26 + 14\times 0.4 + 36\times 0.34 = 7.17$$

$$F(x=1) = \underset{\underline{\omega}}{E}\left[F(x=1,\underline{\omega})\right] = -73\times 0.26 - 18\times 0.4 + 37\times 0.34 = -13.6$$

| *x* | $\omega_1 = 50$ | $\omega_1 = 100$ | $\omega_1 = 150$ | *F(x)* |
|---|---|---|---|---|
| | $p_1 = 0.26$ | $p_1 = 0.40$ | $p_1 = 0.34$ | |
| *x*=1 | 12 | 12 | 12 | 12 |
| *x=2* | -9 | 24 | 24 | 15.42 |
| *x=3* | -41 | 14 | 36 | 7.17 |
| *x=4* | -73 | -18 | 37 | -13.6 |

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# Decision making under uncertainty

The cautious vendor suppresses profits and exaggerates losses

$$\overline{F}(x,\underline{\omega})=\begin{cases}10\times\sqrt{F(x,\underline{\omega})} & if \quad F(x,\underline{\omega})\geq 0\\ -\dfrac{(F(x,\underline{\omega}))^2}{10} & if \quad F(x,\underline{\omega})<0\end{cases}$$

| *x* | $\omega_1 = 50$ | $\omega_2 = 100$ | $\omega_3 = 150$ | *F(x)* |
|---|---|---|---|---|
| | $p_1 = 0.26$ | $p_2 = 0.40$ | $p_3 = 0.34$ | |
| *x*=1 | 34.64 | 34.64 | 34.64 | 34.64 |
| *x*=2 | -8.1 | 48.99 | 48.99 | 34.15 |
| *x*=3 | -168.1 | 37.42 | 60 | -8.34 |
| *x*=4 | -532.9 | -32.4 | 60.83 | -130.83 |

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# Decision making under uncertainty

The „risk-taking” vender exaggerates profits and suppresses losses

$$\overline{F}(x,\underline{\omega}) = \begin{cases} \dfrac{\left(F(x,\underline{\omega})\right)^2}{10} & if \quad F(x,\underline{\omega}) \geq 0 \\ -10 \times \sqrt{-F(x,\underline{\omega})} & if \quad F(x,\underline{\omega}) < 0 \end{cases}$$

| ***x*** | $\omega_1 = 50$ | $\omega_2 = 100$ | $\omega_3 = 150$ | ***F(x)*** |
|---|---|---|---|---|
| | $p_1 = 0.26$ | $p_2 = 0.40$ | $p_3 = 0.34$ | |
| *x*=1 | 14.4 | 14.4 | 14.4 | 14.4 |
| *x=2* | -30 | 57.6 | 57.6 | 34.82 |
| *x=3* | -64.03 | 19.6 | 129.6 | 35.26 |
| *x=4* | -85.44 | -42.43 | 136.9 | 7.36 |

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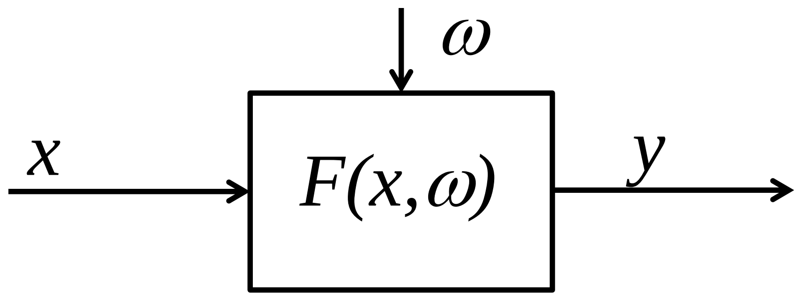
# Decision making under uncertainty




$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right]$$

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$$= \left\{ x \in \mathscr{R}^S;\ \underset{\underline{\omega}}{E}\left[\varphi_l(x,\underline{\omega})\right] = 0,\ l = 1, \dots, L,\ \underset{\underline{\omega}}{E}\left[\psi_m(x,\underline{\omega})\right] \le 0,\ m = 1, \dots, M \right\}$$

$$x^* \to F(x^*) = \min_{x \in D_x} F(x)$$

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A game against nature

# A GAME-THEORETIC APPROACH TO DECISION MAKING UNDER UNCERTAINTY

# A game against nature

A farmer examines the possibility of growing **5. different types of corn.** The size of each grain yield depends on weather conditions. In terms of humidity year may be dry, normal or wet. The table below presents **expected yield** for different weather conditions.

| Type of corn | Weather conditions | | |
|---|---|---|---|
| | drought | normal | rain |
| 1 | 8 | 10 | 12 |
| 2 | 10 | 11 | 7 |
| 3 | 9 | 13 | 8 |
| 4 | 11 | 10 | 6 |
| 5 | 10 | 10 | 9 |

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# A game against nature

**The min – max rule.**
Analyzing the subsequent rows of the matrix we find the maximum revenue that may be achieved for successive states of a nature. We make such a decision, for which the maximum revenue is the smallest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

| Type of corn | Weather conditions | | | max | |
|---|---|---|---|---|---|
| | drought | normal | rain | | |
| 1 | 8 | 10 | 12 | 12 | |
| 2 | 10 | 11 | 7 | 11 | |
| 3 | 9 | 13 | 8 | 13 | |
| 4 | 11 | 10 | 6 | 11 | |
| 5 | 10 | 10 | 9 | 10 | ← min |

# A game against nature

**The WALD's (max – min) rule.**
Analyzing the subsequent rows of the matrix we find the minimum revenue that may be achieved for successive states of a nature. We make such a decision, for which the minimum revenue is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

| Type of corn | Weather conditions | | | min |
|---|---|---|---|---|
| | drought | normal | rain | |
| 1 | 8 | 10 | 12 | 8 |
| 2 | 10 | 11 | 7 | 7 |
| 3 | 9 | 13 | 8 | 8 |
| 4 | 11 | 10 | 6 | 6 |
| 5 | 10 | 10 | 9 | 9 ← max |

# A game against nature

**The max – max rule.**

Analyzing the subsequent rows of the matrix we find the maximum revenue that may be achieved for successive states of a nature. We make such a decision, for which the maximum revenue is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

| Type of corn | Weather conditions | | | max |
|---|---|---|---|---|
| | drought | normal | rain | |
| 1 | 8 | 10 | 12 | 12 |
| 2 | 10 | 11 | 7 | 11 |
| 3 | 9 | 13 | 8 | 13 |
| 4 | 11 | 10 | 6 | 11 |
| 5 | 10 | 10 | 9 | 10 |

← max

# A game against nature

$a_i$ - the minimum profit for *i*–th row

$A_i$ - the maximum profit for *i*–th row

$\gamma$ **- confidence factor**

$$H_i(\gamma) = a_i\,\gamma + A_i(1-\gamma) \quad \gamma \in [0,1]$$

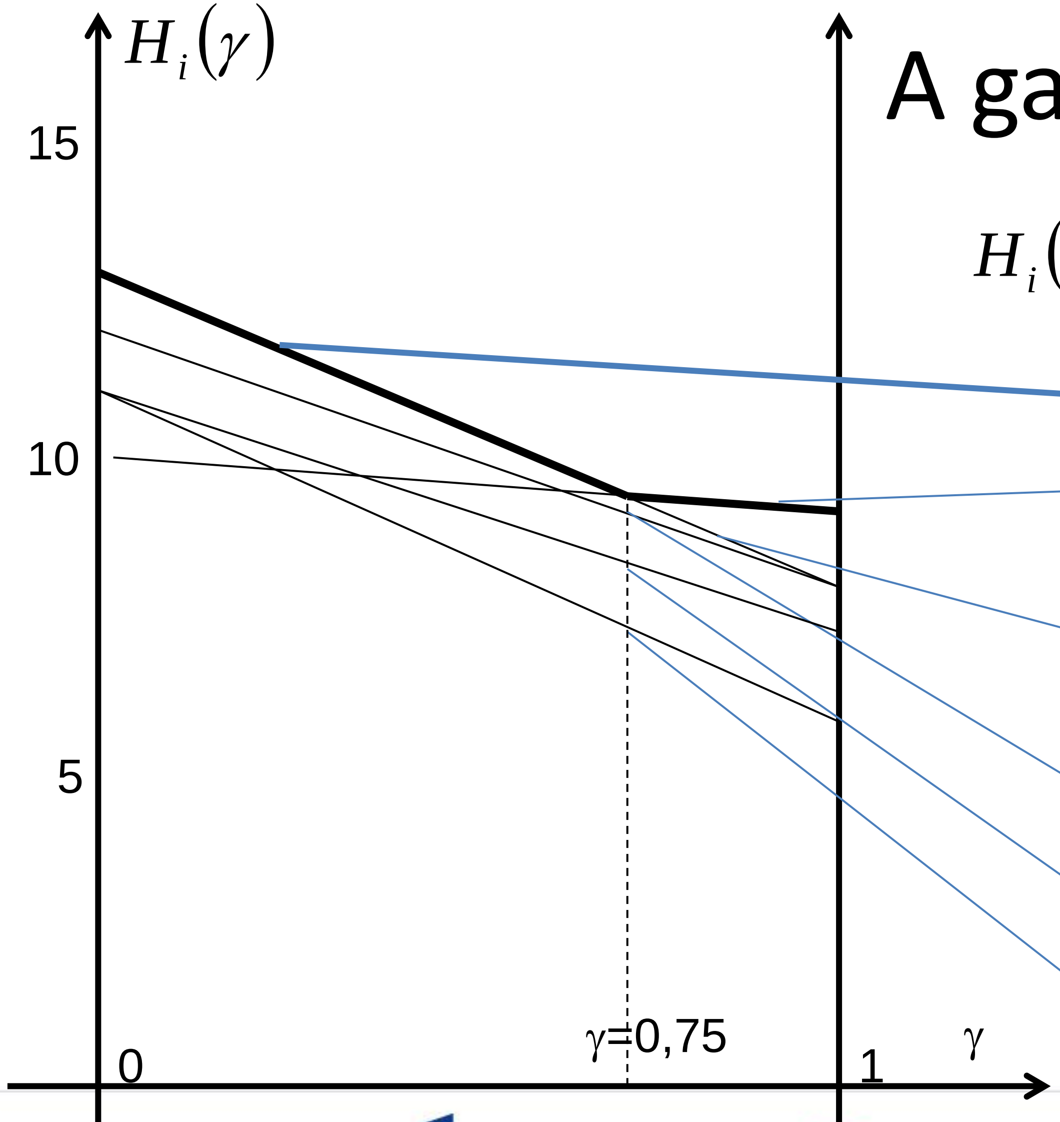
**The Hurwitz rule.**

Analyzing the subsequent rows of the matrix we find the minimum and the maximum revenue, i.e. values $a_i$, $A_i$ and value of the function $H_i(\gamma)$ for a given $\gamma$. We make such a decision, for which the value of the function $H_i(\gamma)$ is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

| Type of corn | Weather conditions | | | *a* min | *A* max | $H(\gamma)$ $\gamma = 0.5$ | |
|---|---|---|---|---|---|---|---|
| | drought | normal | rain | | | | |
| 1 | 8 | 10 | 12 | 8 | 12 | 10 | |
| 2 | 10 | 11 | 7 | 7 | 11 | 9 | |
| 3 | 9 | 13 | 8 | 8 | 13 | 10.5 | ← max |
| 4 | 11 | 10 | 6 | 6 | 11 | 8.5 | |
| 5 | 10 | 10 | 9 | 9 | 10 | 9.5 | |

$\gamma$

# A game against nature




$$H_i(\gamma) = a_i\,\gamma + A_i(1-\gamma) \quad \gamma \in [0,1]$$

$$H(\gamma) = \max_{1 \le i \le 5} \{H_i(\gamma)\}$$

$$H_5(\gamma) = 9\gamma + 10(1-\gamma)$$

$$H_3(\gamma) = 8\gamma + 13(1-\gamma)$$

$$H_1(\gamma) = 8\gamma + 12(1-\gamma)$$

$$H_2(\gamma) = 7\gamma + 11(1-\gamma)$$

$$H_4(\gamma) = 6\gamma + 11(1-\gamma)$$

# A game against nature

**The Laplace rule.**
Analyzing the subsequent rows of the matrix we find the expected revenue, assuming that successive states of nature are equally likely. We make such a decision, for which expected revenue is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

| Type of corn | Weather conditions | | | Expected revenue | |
|---|---|---|---|---|---|
| | drought | normal | rain | | |
| 1 | 8 | 10 | 12 | 30/3 | ← max |
| 2 | 10 | 11 | 7 | 28/3 | |
| 3 | 9 | 13 | 8 | 30/3 | ← max |
| 4 | 11 | 10 | 6 | 27/3 | |
| 5 | 10 | 10 | 9 | 19/3 | |

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Two-person zero-sum game

# A GAME-THEORETIC APPROACH TO DECISION MAKING UNDER UNCERTAINTY

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# Two-person zero-sum game

Two-player zero-sum game

Payoff matrix of player A:

| B / A | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ |
|---|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | … | $a_{1m}$ | … | $a_{1M}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | … | $a_{2m}$ | … | $a_{2M}$ |
| … | … | … | … | … | … | … |
| $A_n$ | $a_{n1}$ | $a_{n2}$ | … | $a_{nm}$ | … | $a_{nM}$ |
| … | … | … | … | … | … | … |
| $A_N$ | $a_{N1}$ | $a_{N2}$ | … | $a_{Nm}$ | … | $a_{NM}$ |

Payoff matrix of player B:

| B / A | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ |
|---|---|---|---|---|---|---|
| $A_1$ | $-a_{11}$ | $-a_{12}$ | … | $-a_{1m}$ | … | $-a_{1M}$ |
| $A_2$ | $-a_{21}$ | $-a_{22}$ | … | $-a_{2m}$ | … | $-a_{2M}$ |
| … | … | … | … | … | … | … |
| $A_n$ | $-a_{n1}$ | $-a_{n2}$ | … | $-a_{nm}$ | … | $-a_{nM}$ |
| … | … | … | … | … | … | … |
| $A_N$ | $-a_{N1}$ | $-a_{N2}$ | … | $-a_{Nm}$ | … | $-a_{NM}$ |

Player A aims to maximize revenue

Player B aims to minimize losses

Usually the payoff matrix of player A is presented

# Example

Two candidates A and B compete for a parliamentary seat in an electoral district. They have to make decision concerning to carry election campaign in the last weekend before the election. Each of candidates can spend one day in the city $M_1$ or $M_2$. They consider (independently) three possible strategies :

$A_1, B_1$ – to spend one day in both cities $M_1$ i $M_2$,

$A_2, B_2$ – to spend two days in $M_1$,

$A_3, B_3$ – to spend two days in $M_2$.

If candidate A chooses strategy $A_1$, and candidate B – accordingly chooses strategies $B_1$, $B_2$ or $B_3$, then candidate A may expect gain of votes by 1%, 2% or 4%.

If candidate A chooses strategy $A_2$, and candidate B – accordingly chooses strategies $B_1$, $B_2$ or $B_3$, then candidate A may expect gain of votes by 1%, 0% or 5%.

If candidate A chooses strategy $A_3$, and candidate B – accordingly chooses strategies $B_1$, $B_2$ or $B_3$, then candidate A may expect gain of votes by 0%, 1% or decrease by 1%.

# Example of payoff matrix

Two-player zero-sum game
Payoff matrix of player A:

| A \ B | | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|
| | $A_1$ | 1 | 2 | 4 |
| | $A_2$ | 1 | 0 | 5 |
| | $A_3$ | 0 | 1 | -1 |

Payoff matrix of player B:

| A \ B | | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|
| | $A_1$ | $-1$ | $-2$ | -4 |
| | $A_2$ | $-1$ | 0 | -5 |
| | $A_3$ | 0 | -1 | 1 |

Player A obtains revenue at the expense of player A and vice versa, hence the sum of payoff matrices of players A and B is zero matrix.

# Two-person zero-sum game

Two-player zero-sum game

Payoff matrix of player A:

| B \ A | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ |
|---|---|---|---|---|---|---|
| $A_1$ | $a_{11}$ | $a_{12}$ | … | $a_{1m}$ | … | $a_{1M}$ |
| $A_2$ | $a_{21}$ | $a_{22}$ | … | $a_{2m}$ | … | $a_{2M}$ |
| … | … | … | … | … | … | … |
| $A_n$ | $a_{n1}$ | $a_{n2}$ | … | $a_{nm}$ | … | $a_{nM}$ |
| … | … | … | … | … | … | … |
| $A_N$ | $a_{N1}$ | $a_{N2}$ | … | $a_{Nm}$ | … | $a_{NM}$ |

Payoff matrix of player B:

| B \ A | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ |
|---|---|---|---|---|---|---|
| $A_1$ | $-a_{11}$ | $-a_{12}$ | … | $-a_{1m}$ | … | $-a_{1M}$ |
| $A_2$ | $-a_{21}$ | $-a_{22}$ | … | $-a_{2m}$ | … | $-a_{2M}$ |
| … | … | … | … | … | … | … |
| $A_n$ | $-a_{n1}$ | $-a_{n2}$ | … | $-a_{nm}$ | … | $-a_{nM}$ |
| … | … | … | … | … | … | … |
| $A_N$ | $-a_{N1}$ | $-a_{N2}$ | … | $-a_{Nm}$ | … | $-a_{NM}$ |

Player A aims to maximize revenue

Player B aims to minimize losses

Usually the payoff matrix of player A is presented

# Decision making using game theory

- Typical approaches to game solving
  - determination of saddle point
  - removal of dominated strategies
  - determination of mixed strategies for:
    - N=2 and M=2
    - N>2 and M>2


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# Decision making using game theory

Two-person zero-sum game

Saddle point: $\max_{1 \le n \le N} \min_{1 \le m \le M} a_{nm} \ne ?= \min_{1 \le n \le N} \max_{1 \le m \le M} a_{nm}$

| A \ B | | | | | | | | min |
|---|---|---|---|---|---|---|---|---|
| | | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ | |
| | $A_1$ | $a_{11}$ | $a_{12}$ | … | $a_{1m}$ | … | $a_{1M}$ | |
| | $A_2$ | $a_{21}$ | $a_{22}$ | … | $a_{2m}$ | … | $a_{2M}$ | |
| | … | … | … | … | … | … | … | |
| | $A_n$ | $a_{n1}$ | $a_{n2}$ | … | $a_{nm}$ | … | $a_{nM}$ | |
| | … | … | … | … | … | … | … | |
| | $A_N$ | $a_{N1}$ | $a_{N2}$ | … | $a_{Nm}$ | … | $a_{NM}$ | |
| max | | | | | | | | |

$\leftarrow \max_{1 \le n \le N} \min_{1 \le m \le M} a_{nm}$

$\uparrow \min_{1 \le n \le N} \max_{1 \le m \le M} a_{nm}$

# Decision making using game theory

Two-person zero-sum game

Saddle point: $\max_{1\le n\le N} \min_{1\le m\le M} a_{nm} = \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$ = 190

| B / A | | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | min |
|---|---|---|---|---|---|---|---|
| | $A_1$ | 180 | 150 | 230 | 170 | 150 | 150 |
| | $A_2$ | 200 | 210 | 220 | 150 | 190 | 150 |
| | $A_3$ | 210 | 230 | 190 | 190 | 200 | 190 |
| | $A_4$ | 150 | 220 | 170 | 180 | 220 | 150 |
| | $A_5$ | 210 | 200 | 160 | 150 | 210 | 150 |
| max | | 210 | 230 | 230 | 190 | 220 | |

$\leftarrow \max_{1\le n\le N} \min_{1\le m\le M} a_{nm}$ (row $A_3$)

$\uparrow \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$ (column $B_4$)

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# Decision making using game theory

Dominant and dominated strategies

Player *A* may choose between strategies: $A_1, A_2, \ldots, A_N$

A strategy $A_{n'}$ is dominated by a dominant strategy $A_{n''}$ if

$$\forall\, m = 1, 2, \ldots M \quad a_{n'm} \le a_{n''m}$$

Player *B* may choose between strategies: $B_1, B_2, \ldots, B_M$

A strategy $B_{m'}$ is dominated by a dominant strategy $B_{m''}$ if

$$\forall\, n = 1, 2, \ldots N \quad a_{nm'} \ge a_{nm''}$$

# Example

Removal of dominated strategies

Step 1.

| A \ B | | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|
| | $A_1$ | 1 | 2 | 4 |
| | $A_2$ | 1 | 0 | 5 |
| | ~~$A_3$~~ | ~~0~~ | ~~1~~ | ~~-1~~ |

$A_1$ ← dominant strategy

$A_3$ ← dominated strategy

Step 2.

| A \ B | | $B_1$ | $B_2$ | ~~$B_3$~~ |
|---|---|---|---|---|
| | $A_1$ | 1 | 2 | ~~4~~ |
| | $A_2$ | 1 | 0 | ~~5~~ |

$B_3$: ↑ - dominated strategy

$B_1$: ↑

↑ - dominant strategy

# Example

Removal of dominated strategies

Step 3.

| A \ B | | $B_1$ | $B_2$ | |
|---|---|---|---|---|
| | $A_1$ | 1 | 2 | ← dominant strategy |
| | ~~$A_2$~~ | ~~1~~ | ~~0~~ | ← dominated strategy |

Step 4.

| A \ B | | $B_1$ | ~~$B_2$~~ |
|---|---|---|---|
| | $A_1$ | **1** | ~~2~~ |
| | | ↑ - dominant strategy | ↑ - dominated strategy |

Result of the game: 1

# Decision making using game theory

Two-player zero-sum game

Mixed strategies : $\max_{1\le n\le N} \min_{1\le m\le M} a_{nm} \neq \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

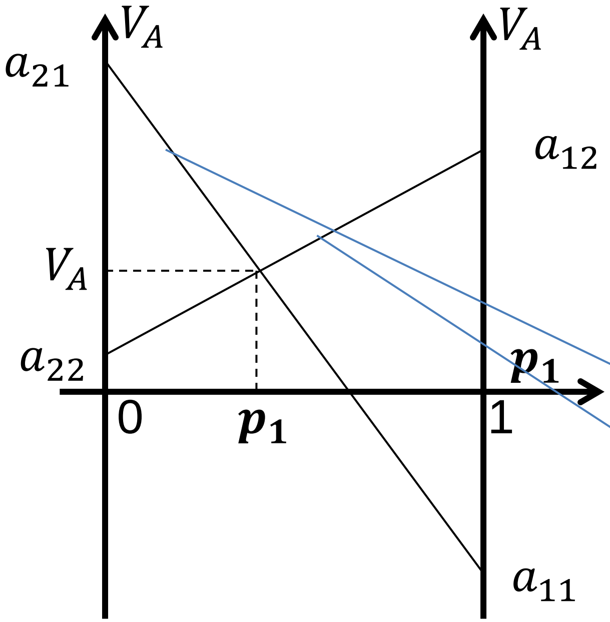
:

| A \ B | | $q_1$ | $q_2$ | … | $q_m$ | … | $q_M$ |
|---|---|---|---|---|---|---|---|
| | | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ |
| $p_1$ | $A_1$ | $a_{11}$ | $a_{12}$ | … | $a_{1m}$ | … | $a_{1M}$ |
| $p_2$ | $A_2$ | $a_{21}$ | $a_{22}$ | … | $a_{2m}$ | … | $a_{2M}$ |
| … | … | … | … | … | … | … | … |
| $p_n$ | $A_n$ | $a_{n1}$ | $a_{n2}$ | … | $a_{nm}$ | … | $a_{nM}$ |
| … | … | … | … | … | … | … | … |
| $p_N$ | $A_N$ | $a_{N1}$ | $a_{N2}$ | … | $a_{Nm}$ | … | $a_{NM}$ |

# Decision making using game theory

*N* = *M* = 2

| B / A | | $q_1$ | $q_2$ |
|---|---|---|---|
| | | $B_1$ | $B_2$ |
| $p_1$ | $A_1$ | $a_{11}$ | $a_{12}$ |
| $p_2$ | $A_2$ | $a_{21}$ | $a_{22}$ |




Equations for player *A*

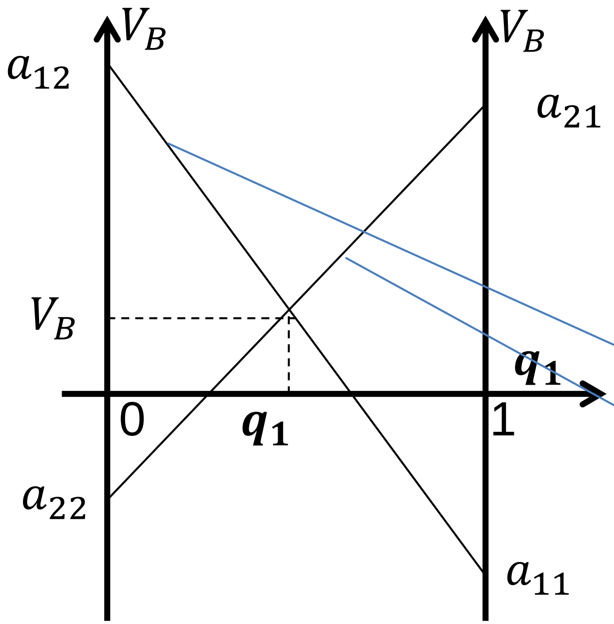
$$p_1 a_{11} + p_2 a_{21} = V_A / B_1$$
$$p_1 a_{12} + p_2 a_{22} = V_A / B_2$$
$$p_1 + p_2 = 1$$

$$p_2 = 1 - p_1$$
$$p_1 a_{11} + (1 - p_1) a_{21} = V_A / B_1$$
$$p_1 a_{12} + (1 - p_1) a_{22} = V_A / B_2$$

# Decision making using game theory

$N = M = 2$

| B \ A | | $q_1$ | $q_2$ |
|---|---|---|---|
| | | $B_1$ | $B_2$ |
| $p_1$ | $A_1$ | $a_{11}$ | $a_{12}$ |
| $p_2$ | $A_2$ | $a_{21}$ | $a_{22}$ |




Equations for player $B$

$$q_1 a_{11} + q_2 a_{12} = V_B \;/\; A_1$$
$$q_1 a_{21} + q_2 a_{22} = V_B \;/\; A_2$$
$$q_1 + q_2 = 1$$

$$q_2 = 1 - q_1$$
$$q_1 a_{11} + (1 - q_1) a_{12} = V_B \;/\; A_1$$
$$q_1 a_{21} + (1 - q_1) a_{22} = V_B \;/\; A_2$$

# Example

Two-person zero-sum game

Saddle point: $\max_{1\le n\le N} \min_{1\le m\le M} a_{nm} \neq \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

| B / A | | $B_1$ | $B_2$ | $B_3$ | min |
|---|---|---|---|---|---|
| | $A_1$ | 3 | $-3$ | 7 | -3 |
| | $A_2$ | $-1$ | 5 | 2 | -1 |
| | $A_3$ | 0 | -4 | 4 | -4 |
| max | | 3 | 5 | 7 | |

$\leftarrow \max_{1\le n\le N} \min_{1\le m\le M} a_{nm}$ (row $A_2$, min = -1)

$\uparrow \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$ (column $B_1$, max = 3)

# Example

Removal of dominated strategies

| A \ B | | $q_1$ | $q_2$ | $q_3$ | |
|---|---|---|---|---|---|
| | | $B_1$ | $B_2$ | $B_3$ | |
| $p_1$ | $A_1$ | 3 | $-3$ | 7 | ← dominant strategy |
| $p_2$ | $A_2$ | $-1$ | 5 | 2 | |
| $p_3$ | $A_3$ | 0 | -4 | 4 | ← dominated strategy |

↑ (under $B_3$) - dominated strategy

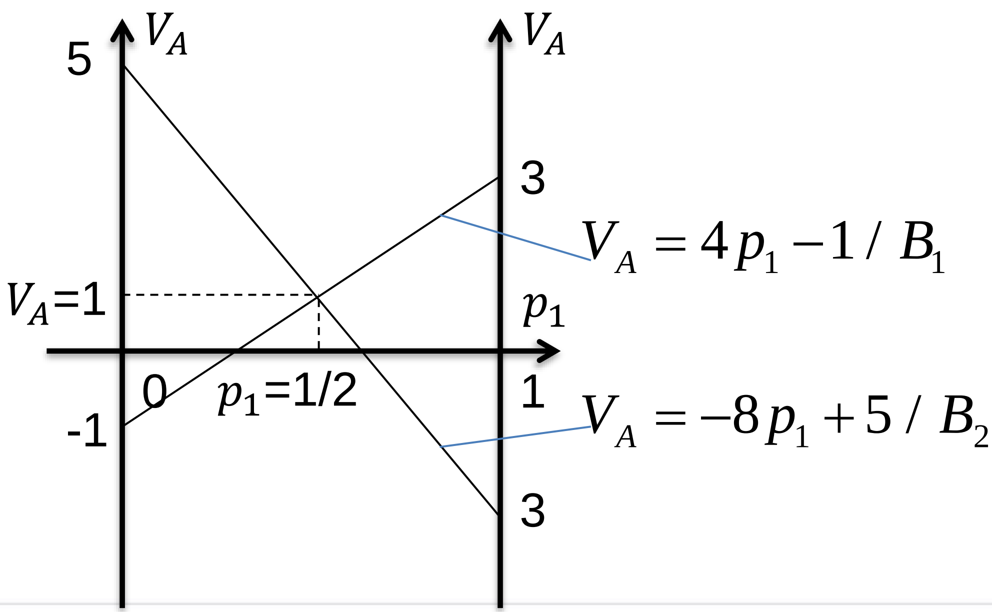
↑ (under $B_1$) - dominant strategy

Since strategies $A_3$ and $B_3$ are dominated $\boldsymbol{p_3}$=$\boldsymbol{q_3}$=0, we need to calculate: $\boldsymbol{p_1}$, $\boldsymbol{p_2}$, $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$

$N = M = 2$

# Example

| A \ B | | $q_1$ | $q_2$ |
|---|---|---|---|
| | | $B_1$ | $B_2$ |
| $p_1$ | $A_1$ | 3 | −3 |
| $p_2$ | $A_2$ | −1 | 5 |




Equations for player *A*

$$3p_1 - p_2 = V_A \,/\, B_1$$

$$-3p_1 + 5p_2 = V_A \,/\, B_2$$

$$p_1 + p_2 = 1$$

$$p_2 = 1 - p_1$$

$$3p_1 - (1 - p_1) = V_A \,/\, B_1$$

$$-3p_1 + 5(1 - p_1) = V_A \,/\, B_2$$

$$p_1 = \frac{1}{2}, \quad p_2 = \frac{1}{2}$$

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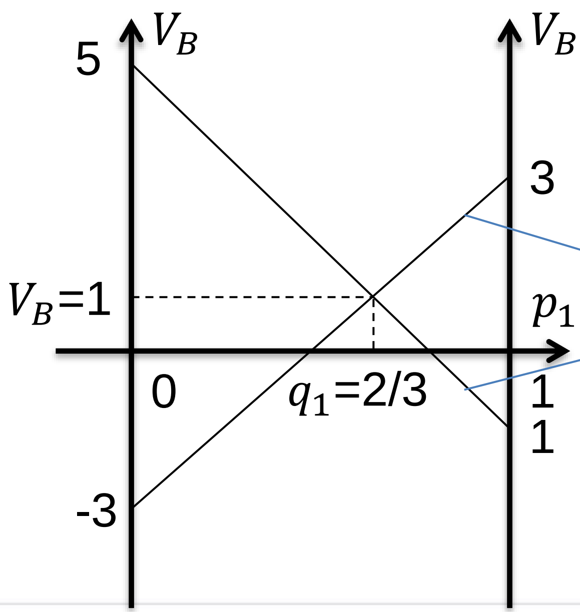
$N = M = 2$

# Example

| A \ B | | $q_1$ | $q_2$ |
|---|---|---|---|
| | | $B_1$ | $B_2$ |
| $p_1$ | $A_1$ | 3 | −3 |
| $p_2$ | $A_2$ | −1 | 5 |




$$V_B = 6q_1 - 3 \;/\; A_1$$

$$V_B = -6q_1 + 5 \;/\; A_2$$

Equations for player *B*

$$3q_1 - 3q_2 = V_B \;/\; A_1$$

$$-q_1 + 5q_2 = V_B \;/\; A_2$$

$$q_1 + q_2 = 1$$

$$q_2 = 1 - q_1$$

$$3q_1 - 3(1 - q_1) = V_B \;/\; A_1$$

$$-q_1 + 5(1 - q_1) = V_B \;/\; A_2$$

$$q_1 = \frac{2}{3}, \quad q_2 = \frac{1}{3}$$

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# Decision making using game theory

Two-player zero-sum game

Mixed strategies *N*>2, *M*>2:

| A \ B | | $q_1$ | $q_2$ | | $q_m$ | | $q_M$ |
|---|---|---|---|---|---|---|---|
| | | $B_1$ | $B_2$ | … | $B_m$ | … | $B_M$ |
| $p_1$ | $A_1$ | $a_{11}$ | $a_{12}$ | … | $a_{1m}$ | … | $a_{1M}$ |
| $p_2$ | $A_2$ | $a_{21}$ | $a_{22}$ | … | $a_{2m}$ | … | $a_{2M}$ |
| … | … | … | … | … | … | … | … |
| $p_n$ | $A_n$ | $a_{n1}$ | $a_{n2}$ | … | $a_{nm}$ | … | $a_{nM}$ |
| … | … | … | … | … | … | … | … |
| $p_N$ | $A_N$ | $a_{N1}$ | $a_{N2}$ | … | $a_{Nm}$ | … | $a_{NM}$ |

In this case solving the game reduces to the linear programming task

# Decision making using game theory

The task for player A:

$$
\left.
\begin{array}{l}
p_1 a_{1m} + p_2 a_{2m} + \cdots + p_N a_{Nm} \geq V_A \; / \; B_m \quad m = 1, 2, \ldots, M \\
p_1 + p_2 + \cdots + p_N = 1 \\
p_n \geq 0 \quad n = 1, 2, \ldots, N
\end{array}
\right| \; /V_A
$$

$$\frac{p_1}{V_A} a_{1m} + \frac{p_2}{V_A} a_{2m} + \cdots + \frac{p_N}{V_A} a_{Nm} \geq 1 \; / \; B_m \quad m = 1, 2, \ldots, M$$

$$\frac{p_1}{V_A} + \frac{p_2}{V_A} + \cdots + \frac{p_N}{V_A} = \frac{1}{V_A}$$

$$\frac{p_n}{V_A} \geq 0 \quad n = 1, 2, \ldots, N$$

Let: $x_n = \dfrac{p_n}{V_A} \quad n = 1, 2, \ldots, N$

# Decision making using game theory

The task for player A

$$a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{Nm}x_N \geq 1 / B_m \quad m = 1, 2, \ldots, M$$

$$x_1 + x_2 + \cdots + x_N = \frac{1}{V_A}$$

Therefore, the expression should be minimized

$$x_n \geq 0 \quad n = 1, 2, \ldots, N$$

Player A aims to maximize the profit

Finally, the task for player A is: $$\min_{x_1, x_2, \ldots, x_N} \left( x_1 + x_2 + \cdots + x_N \right) = V_{A\min}$$

with constraints:

$$a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{Nm}x_N \geq 1 \quad m = 1, 2, \ldots, M$$

$$x_n \geq 0 \quad n = 1, 2, \ldots, N$$

Finally, the solution for player A is: $$p_n = x_n V_{A\min}, \; n = 1, 2, \ldots, N$$

# Decision making using game theory

The task for player B:

$$\left.\begin{array}{l} q_1 a_{n1} + q_2 a_{n2} + \cdots + q_M a_{nM} \le V_B \; / \; A_n \quad n = 1, 2, \ldots, N \\ q_1 + q_2 + \cdots + q_M = 1 \\ q_m \ge 0 \quad m = 1, 2, \ldots, M \end{array}\right| \; /V_B$$

$$\frac{q_1}{V_B} a_{n1} + \frac{q_2}{V_B} a_{n2} + \cdots + \frac{q_M}{V_B} a_{nM} \le 1 \; / \; A_n \quad n = 1, 2, \ldots, N$$

$$\frac{q_1}{V_B} + \frac{q_2}{V_B} + \cdots + \frac{q_M}{V_B} = \frac{1}{V_B}$$

$$\frac{q_m}{V_B} \ge 0 \quad m = 1, 2, \ldots, M$$

Let: $y_m = \frac{q_m}{V_B} \quad m = 1, 2, \ldots, M$

# Decision making using game theory

The task for player B

$$a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nM}y_M \le 1/A_n \quad n = 1, 2, \ldots, N$$

$$y_1 + y_2 + \cdots + y_M = \frac{1}{V_B}$$

Therefore, the expression should be maximized

$$y_m \ge 0 \quad m = 1, 2, \ldots, M$$

Player B minimizes loss

Finally, the task for player B is: $$\max_{y_1, y_2, \ldots, y_M} (y_1 + y_2 + \cdots + y_M) = V_{B\max}$$

with constraints: $$a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nM}y_M \le 1 \quad n = 1, 2, \ldots, N$$

$$y_m \ge 0 \quad m = 1, 2, \ldots, M$$

Finally, the solution for player B is: $$q_m = y_n V_{B\max},\ m = 1, 2, \ldots, M$$

# Example

Two-person zero-sum game

Saddle point: $\max_{1\le n\le N} \min_{1\le m\le M} a_{nm} \neq \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

| A \ B | | $B_1$ | $B_2$ | $B_3$ | min |
|---|---|---|---|---|---|
| | $A_1$ | 5 | 0 | 1 | 0 |
| | $A_2$ | 2 | 4 | 3 | 2 |
| max | | 5 | 4 | 3 | |

$\leftarrow \max_{1\le n\le N} \min_{1\le m\le M} a_{nm}$

$\uparrow \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

# Example

| A \ B | | $q_1$ | $q_2$ | $q_3$ |
|---|---|---|---|---|
| | | $B_1$ | $B_2$ | $B_3$ |
| $p_1$ | $A_1$ | 5 | 0 | 1 |
| $p_2$ | $A_2$ | 2 | 4 | 3 |

The task for player A

$$let\ x_1 = \frac{p_n}{V_A}, n = 1,2$$

$$\min_{x_1, x_2} (x_1 + x_2)$$

Constraints:

$$5x_1 + 2x_2 \geq 1$$
$$0x_1 + 4x_2 \geq 1$$
$$x_1 + 3x_2 \geq 1$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

The task for player B

$$let\ y_m = \frac{q_m}{V_B}, m = 1,2,3$$

$$\max_{y_1, y_2, y_3} (y_1 + y_2 + y_3)$$

Constraints:

$$5y_1 + 0y_2 + y_3 \leq 1$$
$$2y_1 + 4y_2 + 3y_3 \leq 1$$
$$y_1 \geq 0\ y_2 \geq 0\ y_3 \geq 0$$

# Thank you for attention