Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.13b. Identification of dynamic plants

Identification of Dynamic Plants

$u(t)$ → **Identification Plant** → $y(t)$

u_n y_n
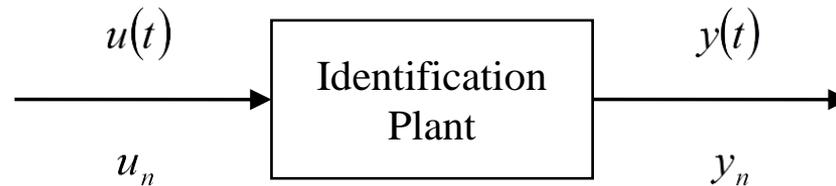
Descriptions:
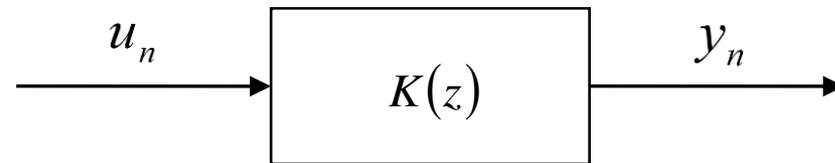
- state variable;
- differential/difference equation;
- transfer functions: $K(s)$, $K(z)$;
- impulse response: $k_i(t)$, k_{in};
- step response: $h(t)$, $h_n(t)$.

Identification of Dynamic Plants
Identification of Impulse Responses

Discrete case

$$y_n = \sum_{k=0}^{n} k_{ik} u_{n-k} = \sum_{k=0}^{n} k_{in-k} u_k \qquad \text{– discrete case}$$

$$k_{in} = \mathscr{Z}^{-1}[K(z)]$$

Identification of Dynamic Plants

For input signal u_n we measure respective output signal y_n :
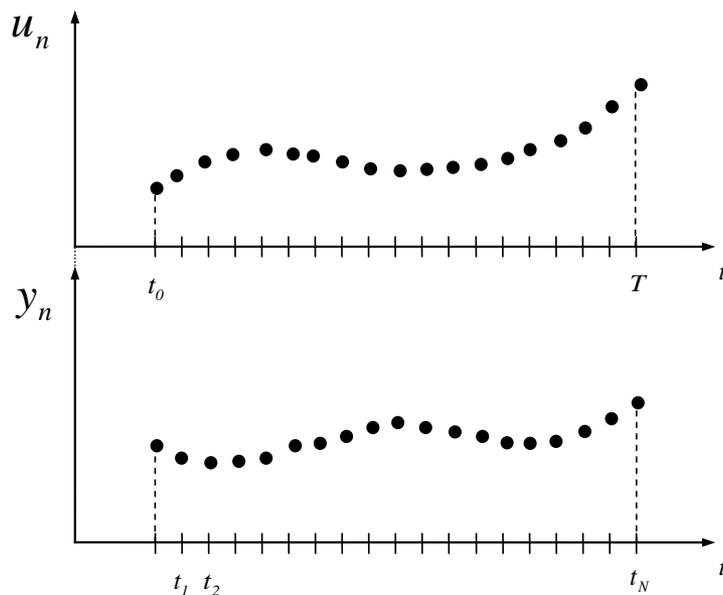
$$t_0 < t_1 < \ldots < t_N \leq T$$

$$\{u_n\}_{n=0}^N \qquad \{y_n\}_{n=0}^N$$

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Identification of Dynamic Plants
Identification of Impulse Responses

Discrete case

We want to determine discrete input response (for $u_n = \delta_n$): k_{in}

We have sequence of measurements: $u_0, u_1, \ldots, u_N$
$y_0, y_1, \ldots, y_N$
and following tasks:

1) plant is linear: $k_{in}(\theta)$, but we don't know the values of parameters θ

2) we want to determine sequence of values of $k_{i0}, k_{i1}, \ldots, k_{iN}$

where:

k_{in} - impulse function

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Identification of Dynamic Plants
Identification of Impulse Responses

Discrete case

3) plant is nonlinear, we approximate: $\bar{k}_{in}(\theta), \ \theta^*$

4) plant is nonlinear, we approximate by the sequence of discrete impulse

response $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

where:

$\bar{k}_{in}$ - given function

θ - unknown vector of parameter

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 1. Linear plant

$$y_n = \sum_{k=0}^{N} k_{ik}(\theta) u_{n-k}$$

where:

k_{ik} - given function, θ - unknown

$$y_0 = k_{i0}(\theta) u_0$$
$$y_1 = k_{i0}(\theta) u_1 + k_{i1}(\theta) u_0$$
$$\vdots$$
$$y_N = k_{i0}(\theta) u_N + k_{i1}(\theta) u_{N-1} + \ldots + k_{iN}(\theta) u_0$$

$N \cdot L \geq R$

$\dim y = L$

$\dim \theta = R$

Solution of the above system of equations is calculated values of θ

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 2. Determination of discrete impulse responses values $k_{i0}, k_{i1}, \ldots, k_{iN}$

$$y_n = \sum_{k=0}^{n} k_{ik} u_{n-k}$$

$$\begin{cases} y_0 = k_{i0} u_0 \\ y_1 = k_{i0} u_1 + k_{i1} u_0 \\ \quad \vdots \\ y_N = k_{i0} u_N + k_{i1} u_{N-1} + \ldots + k_{iN} u_0 \end{cases}$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 2. Determination of discrete impulse responses values $k_{i0}, k_{i1}, \ldots, k_{iN}$

Denote:

$$
\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} = Y_N
\qquad
\begin{bmatrix} u_0 & 0 & \cdots & 0 \\ u_1 & u_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ u_N & u_{N-1} & \cdots & u_0 \end{bmatrix} = U_N
\qquad
\begin{bmatrix} k_{i0} \\ k_{i1} \\ \vdots \\ k_{iN} \end{bmatrix} = \mathbf{k}_{iN}
$$

Solution:

$$
Y_N = U_N \mathbf{k}_{in} \Rightarrow \underline{\underline{\mathbf{k}_{in} = U_N^{-1} Y_N}}
$$

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Identification of Dynamic Plants
Identification of Impulse Responses

Ad. 3. Plant is nonlinear; approximation by $\bar{k}_{in}(\theta)$.

$$\bar{y}_n = \sum_{k=0}^{N} \bar{k}_{ik}(\theta) u_{n-k} \quad - \text{approximation}$$

where: $\bar{k}_{ik}$ - given function, θ - unknown vector of parameters

Performance index:

$$Q_N(\theta) = \sum_{n=0}^{N} (y_n - \bar{y}_n)^2 = \sum_{n=0}^{N} \left(y_n - \sum_{k=0}^{n} \bar{k}_{ik}(\theta) u_{n-k} \right)^2$$

Optimization problem:

$$\theta_N^* \rightarrow Q_N(\theta_N^*) = \min_{\theta \in \Theta} Q_N(\theta)$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence of discrete impulse responses $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

Model:

$$\bar{y}_n = \sum_{k=0}^{N} \bar{k}_{ik} u_{n-k}$$

Performance index:

$$Q_N\left(\bar{k}_{i0}, \bar{k}_{i1}, \ldots, \bar{k}_{iN}\right) = \sum_{n=0}^{N} \left(y_n - \bar{y}_n\right)^2 = \sum_{n=0}^{N} \left(y_n - \sum_{k=0}^{n} \bar{k}_{ik} u_{n-k} \right)^2$$

Optimization problem:

$$k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^* \to Q_N\left(k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*\right) = \min_{\bar{k}_{i0}, \ldots, \bar{k}_{iN}} Q_N\left(\bar{k}_{i0}, \bar{k}_{i1}, \ldots, \bar{k}_{iN}\right)$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^{*}, k_{i1}^{*}, \ldots, k_{iN}^{*}$

Solution:

$$\left. \frac{\partial Q_N\left(\bar{k}_{i0}, \bar{k}_{i1}, \ldots, \bar{k}_{iN}\right)}{\partial \bar{k}_{ip}} \right|_{\substack{\bar{k}_{i0} = \bar{k}_{i0}^{*} \\ \bar{k}_{i1} = \bar{k}_{i1}^{*} \\ \vdots \\ \bar{k}_{iN} = \bar{k}_{iN}^{*}}} = 0 \qquad p = 0, 1, 2, \ldots, N$$

$$-2 \sum_{n=0}^{N} \left(y_n - \sum_{k=0}^{n} k_{ik}^{*} u_{n-k} \right) u_{n-p} = 0 \qquad p = 0, 1, 2, \ldots, N$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k^*_{i0}, k^*_{i1}, \ldots, k^*_{iN}$

$$\sum_{n=0}^{N} y_n u_{n-p} = \sum_{k=0}^{N} \sum_{n=0}^{k} k^*_{ik} u_{n-k} u_{n-p} \qquad p = 0,1,2,\ldots,N$$

Since for $n < 0$ equality $u_n = 0$ holds, we can replace n with N

$$\sum_{n=0}^{N} y_n u_{n-p} = \sum_{k=0}^{N} k^*_{ik} \sum_{n=0}^{N} u_{n-k} u_{n-p} \qquad p = 0,1,2,\ldots,N$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

Let us denote $n - p = \chi$. Then we can rewrite the set of equations:

$$\sum_{\chi=0}^{N-p} y_{\chi+p} u_\chi = \sum_{k=0}^{N} k_{ik}^* \sum_{\chi=0}^{N-p} u_{\chi+p-k} u_\chi \qquad p = 0,1,2,\ldots,N$$

where:

$$\sum_{\chi=0}^{N-p} y_{\chi+p} u_\chi = \rho_{yu,p}[p] \qquad \text{– correlation of input and output signal}$$

$$\sum_{\chi=0}^{N-p} u_{\chi+p-k} u_\chi = \rho_{uu,p}[p-k] \qquad \text{– autocorrelation of input signal}$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

The set of equations can be expressed in the equivalent form:

$$\rho_{yu,p}[p] = \sum_{k=0}^{N} k_{ik}^* \rho_{uu,p}[p-k] \qquad p = 0,1,2,\ldots,N$$

or

$$\begin{bmatrix} \rho_{yu,0}[0] \\ \rho_{yu,1}[1] \\ \vdots \\ \rho_{yu,N}[N] \end{bmatrix} = \begin{bmatrix} \rho_{uu,0}[0] & \rho_{uu,0}[-1] & \cdots & \rho_{uu,0}[-N] \\ \rho_{uu,1}[1] & \rho_{uu,1}[0] & \cdots & \rho_{uu,1}[1-N] \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{uu,N}[N] & \rho_{uu,N}[N-1] & \cdots & \rho_{uu,N}[0] \end{bmatrix} \begin{bmatrix} k_{i0}^* \\ k_{i1}^* \\ \vdots \\ k_{iN}^* \end{bmatrix}$$
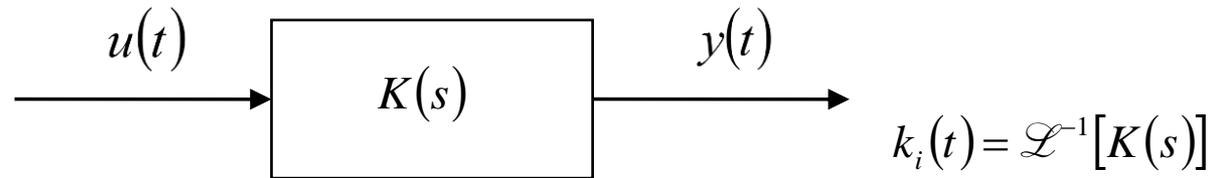
$$\overline{\rho}_{yu} = \overline{\rho}_{uu} k_{iN}^*$$
$$\Downarrow$$
$$k_{iN}^* = \rho_{uu}^{-1} \rho_{yu}$$

Identification of Dynamic Plants
Identification of Impulse Responses

Continuous case

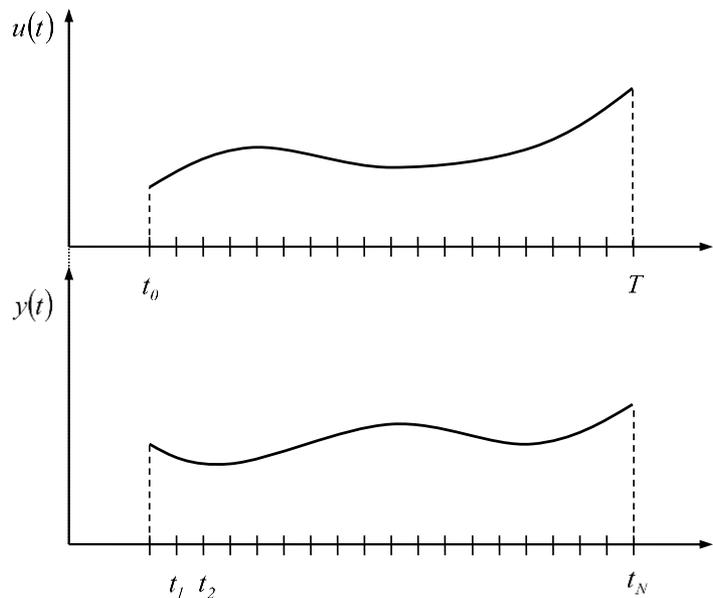$$u(t) \rightarrow \boxed{K(s)} \rightarrow y(t)$$

$$k_i(t) = \mathscr{L}^{-1}[K(s)]$$

$$y(t) = \int_0^T k_i(t-\tau)u(\tau)d\tau = \int_0^T k_i(\tau)u(t-\tau)d\tau \quad - \text{continuous case}$$

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Identification of Dynamic Plants

For input signal $\{u(t)\}_{t_0}^T$ we measure respective output signal $\{y(t)\}_{t_0}^T$:

$$t_0 < t_1 < \ldots < t_N \leq T$$

$$\{u(t_n)\}_{n=0}^N \qquad \{y(t_n)\}_{n=0}^N$$

Identification of Dynamic Plants
Identification of Impulse Responses

Continuous Case

We have measurements: $\{u(t)\}_{t=0}^{T}$, $\{y(t)\}_{t=0}^{T}$

and following tasks:

1) plant is linear and we don't know the values of parameters: $k_i(t,\theta)$, θ

2) we want to determine $k_i(t)$

where:

$k_i(t,\theta)$ - known function, θ - unknown vector of parameters

Identification of Dynamic Plants
Identification of Impulse Responses

Continuous Case

3) plant is nonlinear, we approximate: $\bar{k}_i(t,\theta)$, θ

4) plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

where:

$\bar{k}_i(t,\theta)$ - given function, θ - unknown vector of parameters

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 1. plant is linear and we don't know the values of parameters: $k_i(t,\theta)$, θ

$$y(t) = \int_o^t k_i(\tau,\theta)u(t-\tau)d\tau$$

$$y(t_n) = \int_o^{t_n} k_i(\tau,\theta)u(t-\tau)d\tau \qquad n = 1,2,\ldots,R$$

Solution of the above system of equations with respect θ gives identification algorithm.

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 2. we want to determine $k_i(t)$

$$k_i(t) = \mathscr{L}^{-1}[K(s)]$$

$$K(s) = \frac{\mathscr{L}[y(t)]}{\mathscr{L}[u(t)]}$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 3. Plant is nonlinear, we approximate: $\bar{k}_i(t,\theta), \theta$

where:

Model:
$$\bar{y}(t,\theta) = \int_o^t \bar{k}_i(\tau,\theta) u(t-\tau) d\tau$$

Approximation:

Performance index: $\quad Q_T(\theta) = \int_0^T (y(t) - \bar{y}(t,\theta))^2 \, dt = \int_0^T \left(y(t) - \int_o^t k_i(\tau,\theta) u(t-\tau) d\tau \right)^2 dt$

Optimization problem: $\qquad \theta_T^* \to Q_T(\theta^*) = \min_\theta Q_T(\theta)$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

Model:

$$\bar{y}(t) = \int\limits_{o}^{t} \bar{k}_i(\tau)u(t-\tau)d\tau$$

Performance index:

$$Q\left(\bar{k}_i(\tau)\right) = \int\limits_{0}^{T}\left(y(t)-\bar{y}(t)\right)^2 dt = \int\limits_{0}^{T}\left(y(t)-\int\limits_{o}^{t}\bar{k}_i(\tau)u(t-\tau)d\tau\right)^2 dt$$

Optimization task (minimization of functional):

$$k_i^*(\tau) \to Q_T\left(k_i^*(\tau)\right) = \min_{\bar{k}_i(\tau)} Q_T\left(\bar{k}_i(\tau)\right)$$
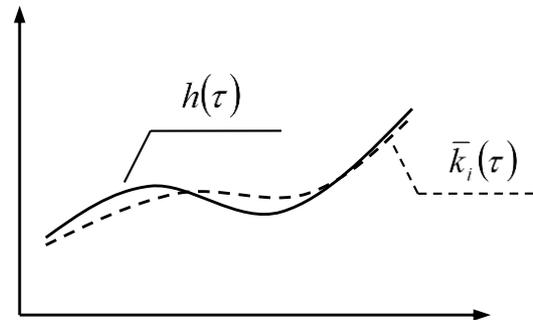
Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$k_i^*(\tau) \rightarrow Q_T\left(k_i^*(\tau)\right) = \min_{\bar{k}_i(\tau)} Q_T\left(\bar{k}_i(\tau)\right)$$

$\bar{k}_i(\tau) + \delta h(\tau)$

$$\left.\frac{\partial Q_T}{\partial \delta}\right|_{\delta=0} = 0, \qquad \forall h(\tau)$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$Q\left(\bar{k}_i(\tau)+\delta h(\tau)\right)=\int_0^T\left(y(t)-\int_0^t\left(\bar{k}_i(\tau)+\delta h(\tau)\right)u(t-\tau)d\tau\right)^2 dt =$$

$$=\int_0^T\left(y(t)-\int_0^t\bar{k}_i(\tau)u(t-\tau)d\tau-\delta\int_0^t h(\tau)u(t-\tau)d\tau\right)^2 dt=\int_o^T\left(y(t)-\int_0^t\bar{k}_i(\tau)u(t-\tau)d\tau\right)^2 dt +$$

$$-2\delta\int_0^T\left(y(t)-\int_0^t\bar{k}_i(\tau)u(t-\tau)d\tau\right)\int_0^t h(\chi)u(t-\chi)d\chi dt+\delta^2\int_0^T\left(\int_0^t h(\tau)u(t-\tau)d\tau\right)^2 dt =$$

$$=Q_1-2\delta Q_2+\delta^2 Q_3 = Q$$

$$\frac{\partial Q}{\partial \delta}=-2Q_2+2\delta Q_3\big|_{\delta=0}=0 \;\Rightarrow\; Q_2=0 \quad \forall h(\tau)$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\forall h(\tau) \quad \int_0^T \left(y(t) - \int_0^t k_i^*(\tau)u(t-\tau)d\tau \right) \int_0^t h(\chi)u(t-\chi)d\chi\, dt = 0$$

$u(t)=0$ for $t<0$, so $t \to T$

$$\forall h(\tau) \quad \int_0^T \left(y(t) - \int_0^T k_i^*(\tau)u(t-\tau)d\tau \right) \int_0^T h(\chi)u(t-\chi)d\chi\, dt = 0$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\forall h(\tau) \quad \int_0^T h(\chi)\left(\int y(t)u(t-\chi)dt - \int_0^T k_i^*(\tau)\int_o^T u(t-\tau)u(t-\chi)dtd\tau \right)d\chi = 0$$

$$\Downarrow$$
$$0$$

$$\downarrow$$

$$\int_o^T y(t)u(t-\chi)dt = \int_0^T k_i^*(\tau)\int_o^T u(t-\tau)u(t-\chi)dtd\tau$$

where: $t - \chi = \gamma$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\int_o^{T-\gamma} y(\gamma+\chi)u(\gamma)dt = \int_0^T k_i^*(\tau)\int_o^{T-\gamma} u(\gamma)u(\gamma+\chi-\tau)dtd\tau$$

$$\rho_{yu}(\chi) = \int_0^T k_i^*(\tau)\rho_{uu}(\chi-\tau)d\tau$$

Thank you for attention

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