Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.16 Summary

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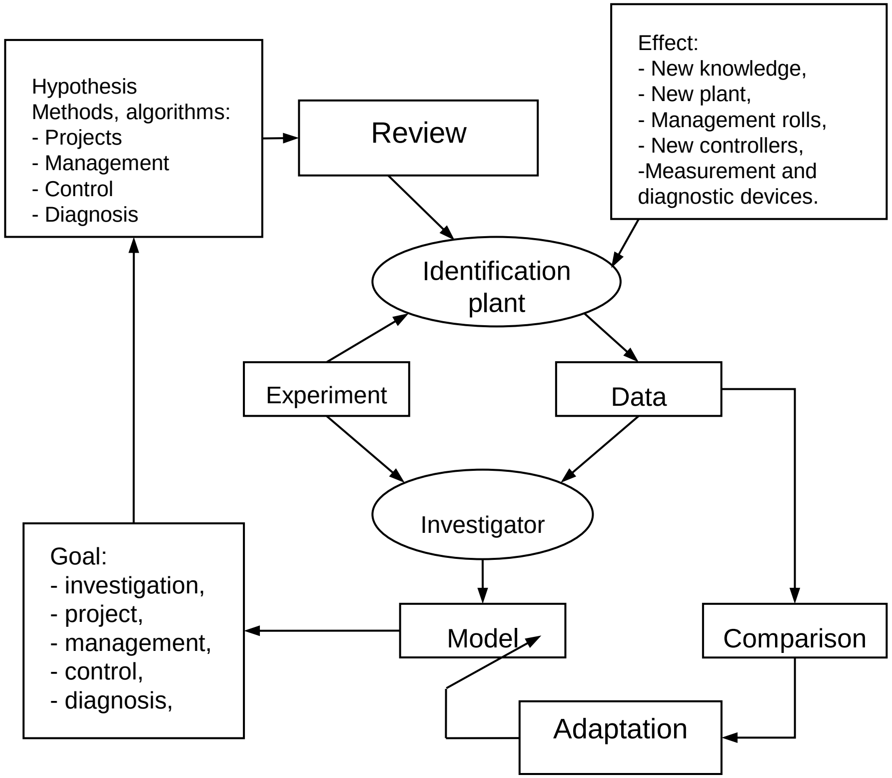
Model in the systems research



Hypothesis
Methods, algorithms:
- Projects
- Management
- Control
- Diagnosis

Review

Effect:
- New knowledge,
- New plant,
- Management rolls,
- New controllers,
-Measurement and diagnostic devices.

Identification plant

Experiment

Data

Investigator

Goal:
- investigation,
- project,
- management,
- control,
- diagnosis,

Model

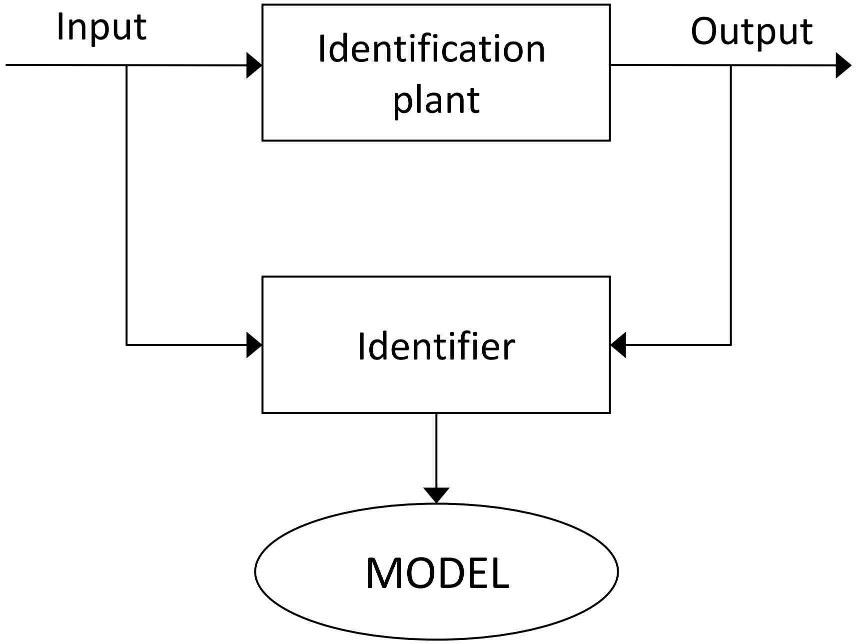
Comparison

Adaptation

Model in the systems research

- Conceptual models
- Physical models
- Analog models
- Mathematical models
- Computer models

Identification Task



Input

Identification plant

Output

Identifier

MODEL

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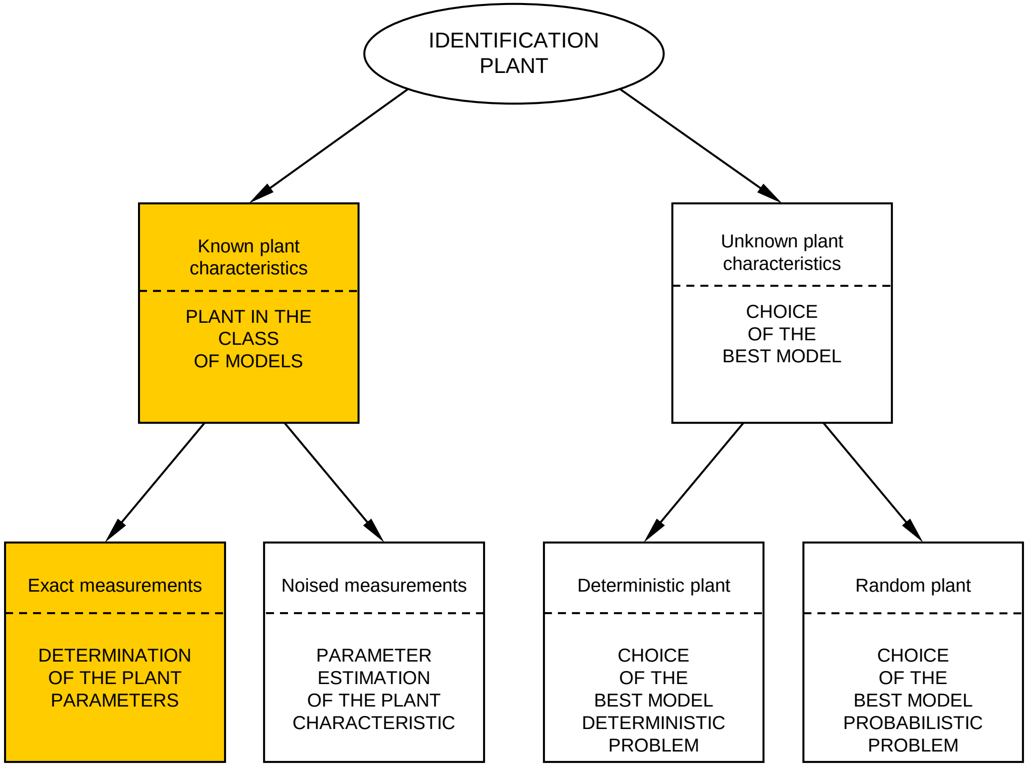
Identification task

1. Determination of the identification plant
2. Determination of the class model
3. Experiment organization
4. Determination of the identification algorithms
5. Identifiers realization

Typical identification tasks



IDENTIFICATION
PLANT

Known plant
characteristics

PLANT IN THE
CLASS
OF MODELS

Unknown plant
characteristics

CHOICE
OF THE
BEST MODEL

Exact measurements

DETERMINATION
OF THE PLANT
PARAMETERS

Noised measurements

PARAMETER
ESTIMATION
OF THE PLANT
CHARACTERISTIC

Deterministic plant

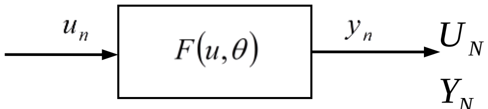
CHOICE
OF THE
BEST MODEL
DETERMINISTIC
PROBLEM

Random plant

CHOICE
OF THE
BEST MODEL
PROBABILISTIC
PROBLEM

Plant in the class of model

Deterministyczny

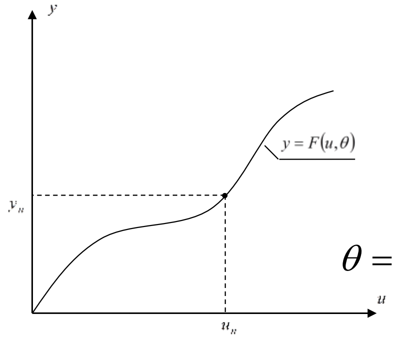


$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

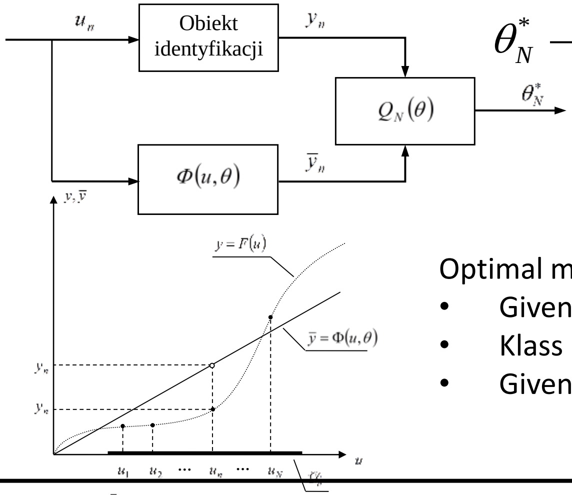
$$y_n = F(u_n,\theta),$$
$$n = 1, 2, \ldots, N$$

$$Y_N = \overline{F}(U_N,\theta)$$

$$\theta = \overline{F}^{-1}(U_N,Y_N) \overset{df}{=} \Psi_N(U_N,Y_N)$$



Choice of the best model



$$\bar{y} = \Phi(u,\theta_N^*)$$

$$\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$$

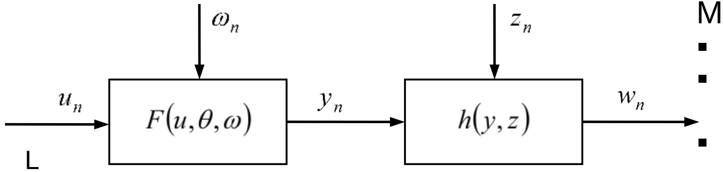
$$\theta_N^* = \Psi_N(U_N,Y_N)$$

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

Optimal model for:

- Given measurements sequences
- Klass of models
- Given performance index

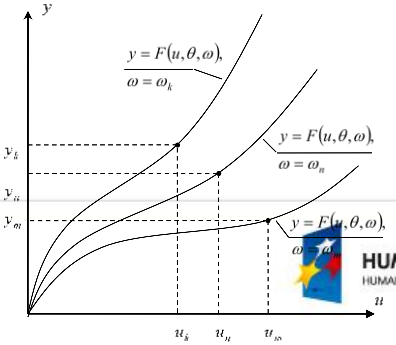
Losowy

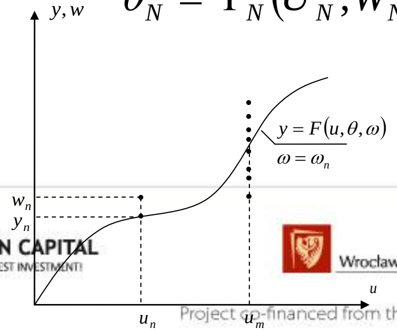


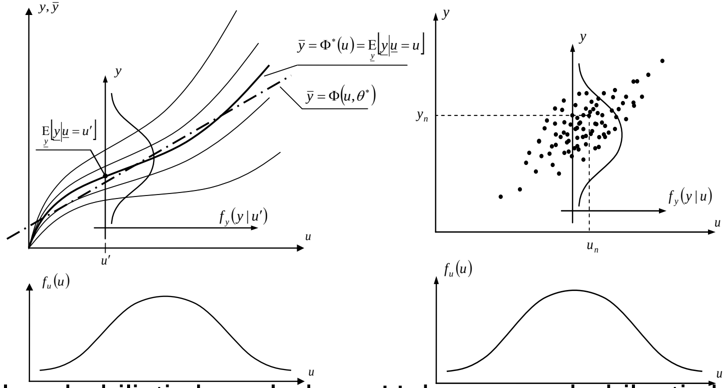
Methods:

- Least square
- Maximum likelihood
- Bayes'a

$$\theta_N = \Psi_N(U_N,W_N)$$







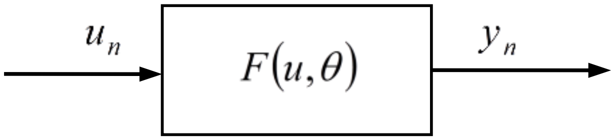
Full probabilistic knowledge

- First type regrsesion
- Second type regression

Unknown probabilostic knowledge

- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

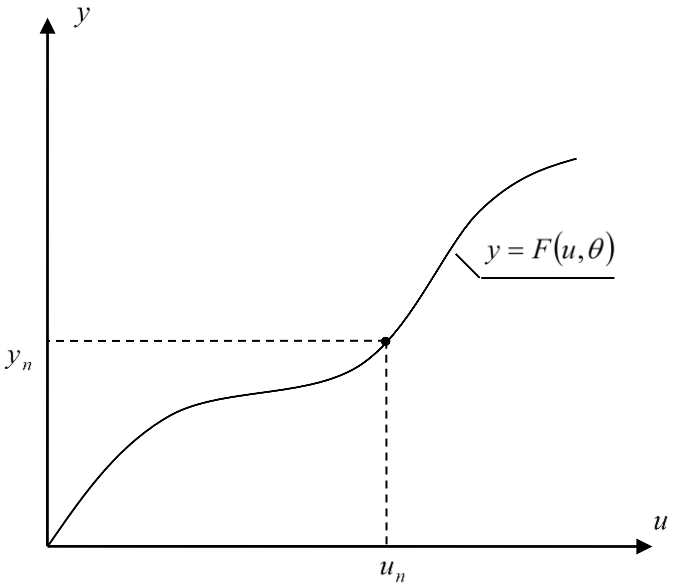
Determination of the plant parameters (3)



Measurements:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N],$$

$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$



Determination of the plant parameters (4)

System of equations:

$$y_n = F(u_n, \theta), \quad n = 1, 2, \ldots, N$$

can be written

$$[y_1 \quad y_2 \quad \cdots \quad y_N] = [F(u_1, \theta) \quad F(u_2, \theta) \quad \cdots \quad F(u_N, \theta)]$$

For

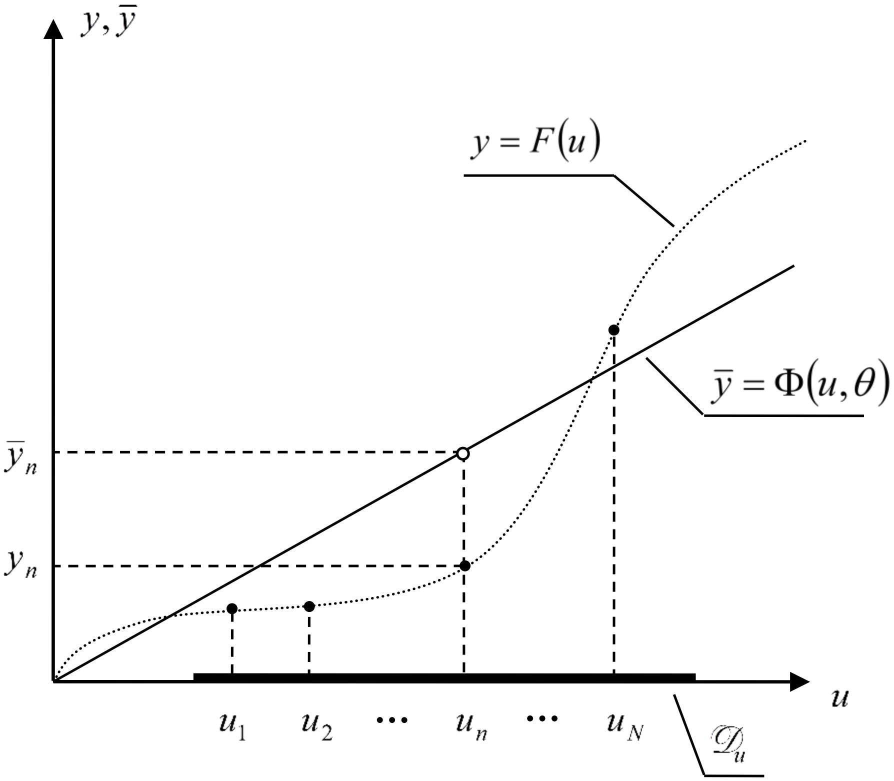
$$[F(u_1, \theta) \quad F(u_2, \theta) \quad \cdots \quad F(u_N, \theta)] \overset{df}{=} \overline{F}(U_N, \theta)$$

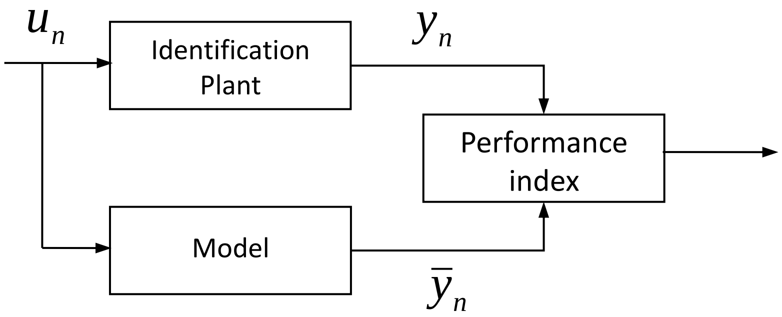
we can rewrite given set of equations:

$$Y_N = \overline{F}(U_N, \theta)$$

Choice of the best model

Deterministic problem





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Choice of the best model based on the noise free measurements

- Problem formulation

<u>Performance index:</u> $Q_N(\theta)=\left\|Y_N-\bar{Y}_N(\theta)\right\|_{U_N}$

where: $\bar{Y}_N(\theta)\overset{df}{=}\left[\Phi(u_1,\theta)\quad\Phi(u_2,\theta)\quad\cdots\quad\Phi(u_N,\theta)\right]$

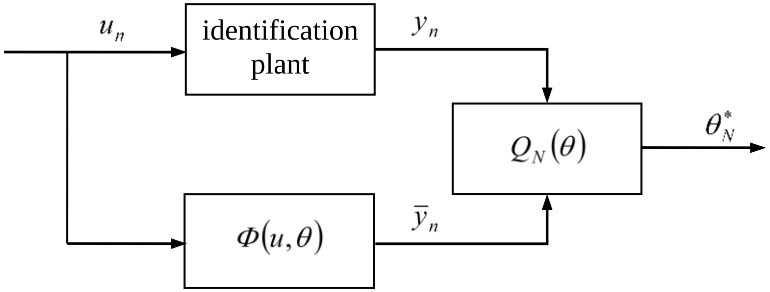
$Q_N(\theta)=\sum_{n=1}^{N}\alpha_n q(y_n,\bar{y}_n)=\sum_{n=1}^{N}\alpha_n q(y_n,\Phi(u_n,\theta))$ e. g. : $Q_N(\theta)=\sum_{n=1}^{N}|y_n-\bar{y}_n|=\sum_{n=1}^{N}|y_n-\Phi(u_n,\theta)|$

$Q_N(\theta)=\max_{1\le n\le N}\{q(y_n,\bar{y}_n)\}=\max_{1\le n\le N}\{q(y_n,\Phi(u_n,\theta))\}$ e. g. : $Q_N(\theta)=\max_{1\le n\le N}\{|y_n-\bar{y}_n|\}=\max_{1\le n\le N}\{|y_n-\Phi(u_n,\theta)|\}$

Choice of the best model based on the noise free measurements

- Problem formulation

Optimal model: $\bar{y} = \Phi(u, \theta_N^*)$ $\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta} Q_N(\theta)$



The model is optimal for:

- given measurement sequence
- proposed model
- performance index

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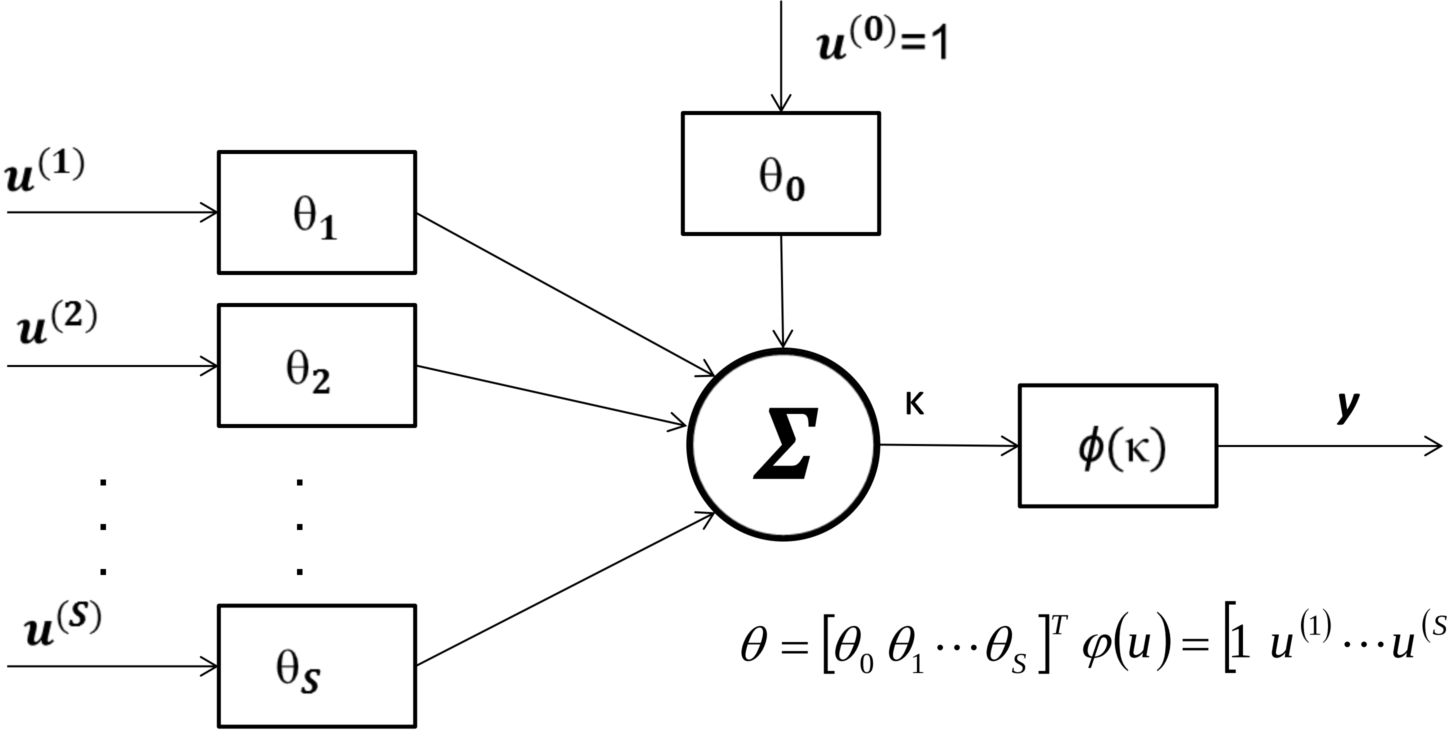
Neuron model simplification



$$\theta = [\theta_0 \; \theta_1 \cdots \theta_S]^T \; \varphi(u) = [1 \; u^{(1)} \cdots u^{(S)}]^T$$

$$y = \phi\left(\sum_{s=1}^{S} \theta_s u^{(s)} + \theta_0\right) = \phi\left(\theta^T \varphi(u)\right)$$

Φ – activation function

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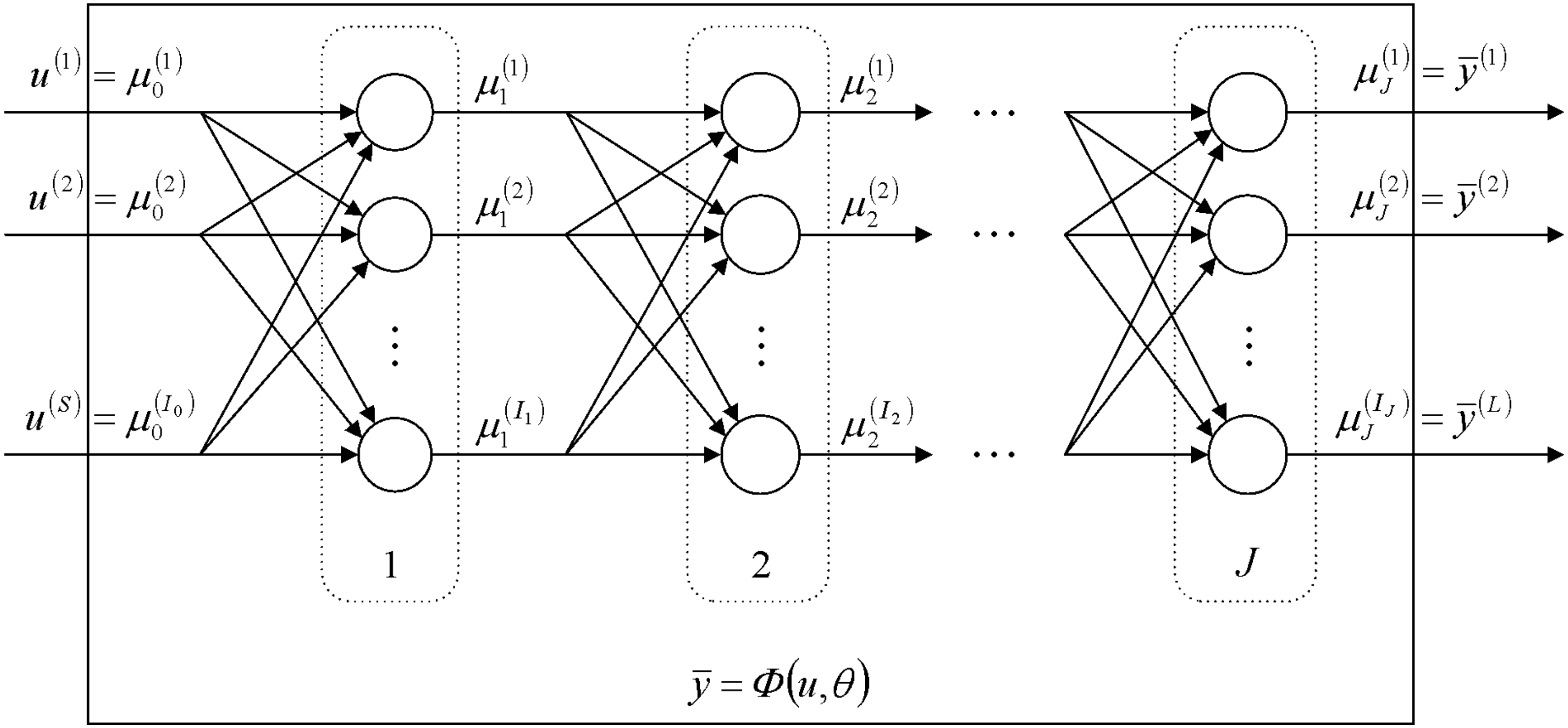
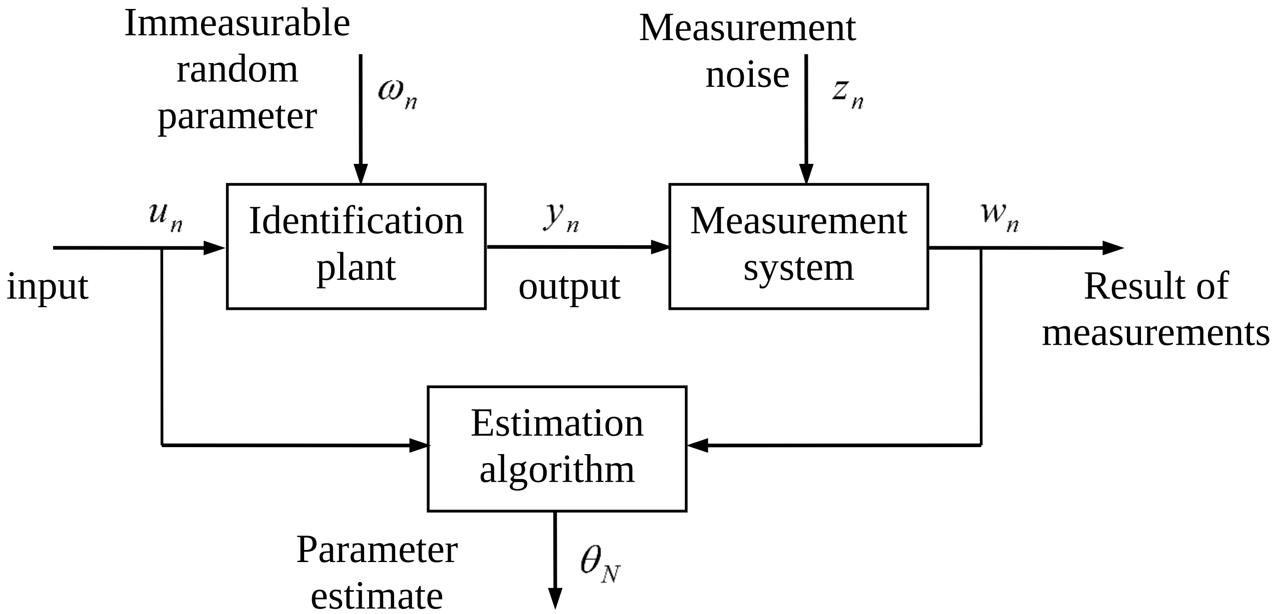
Multilayer network

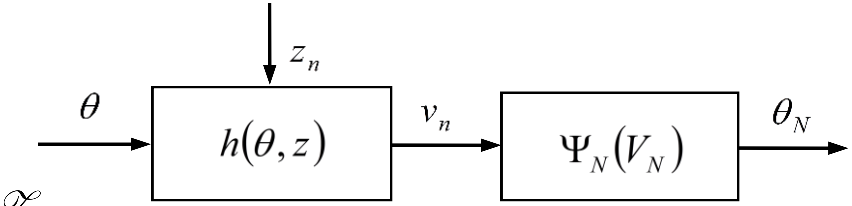


Plant parameter estimation problem



Noised measurements of the physical values

- Problem formulation



Measurement noise:

z_n – value of random variable $\underline{z}$ from the space $\mathscr{Z}$

$f_z(z)$ – probability density function

θ – observed vector of parameters, value of random variable $\underline{\theta}$, $\theta \in \Theta \subseteq \mathscr{R}^R$

$f_\theta(\theta)$ – probability density function

Measurements: $V_N = [v_1 \quad v_2 \quad \cdots \quad v_N]$

Noised measurements of the physical values

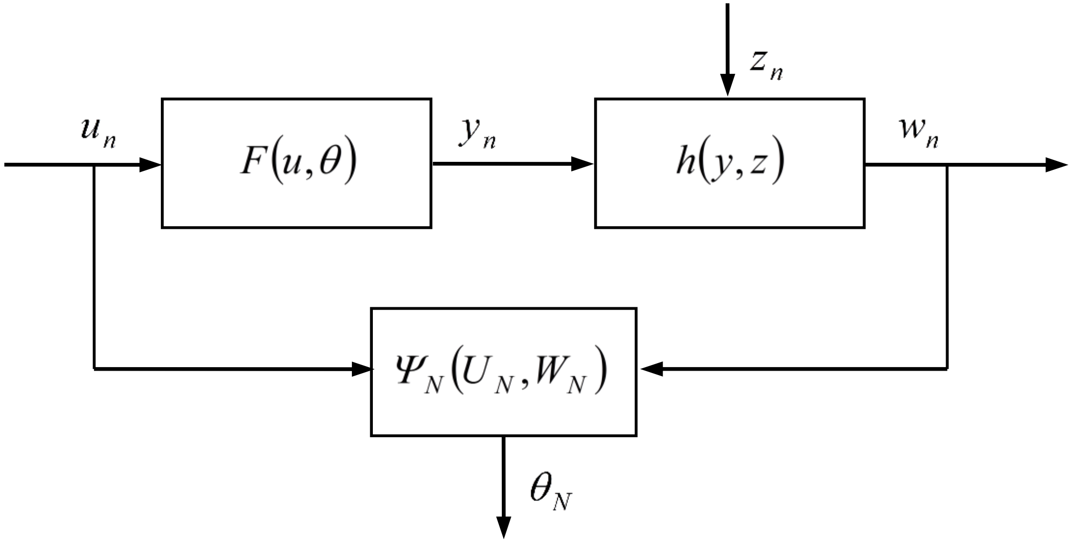
General form of estimation algorithm:

$$\theta_N = \Psi_N(V_N)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayesian method

Plant parameter estimation problem

- Deterministic plant, noised measurements of the plant output



where:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$$

$$W_N = [w_1 \quad w_2 \quad \cdots \quad w_N]$$

Ψ_N – estimation algorithm

θ_N – estimate of θ

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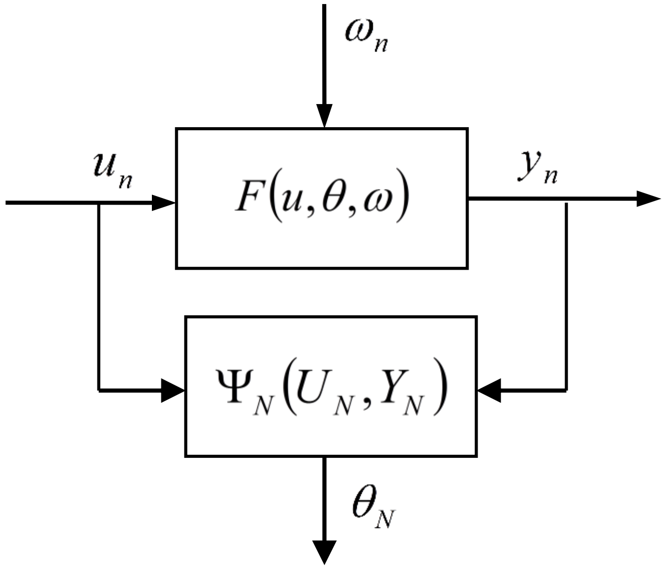
Plant parameter estimation problem

- Immeasurable random plant parameter



where:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$$

$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

Ψ_N – estimation algorithm

θ_N – estimate of θ

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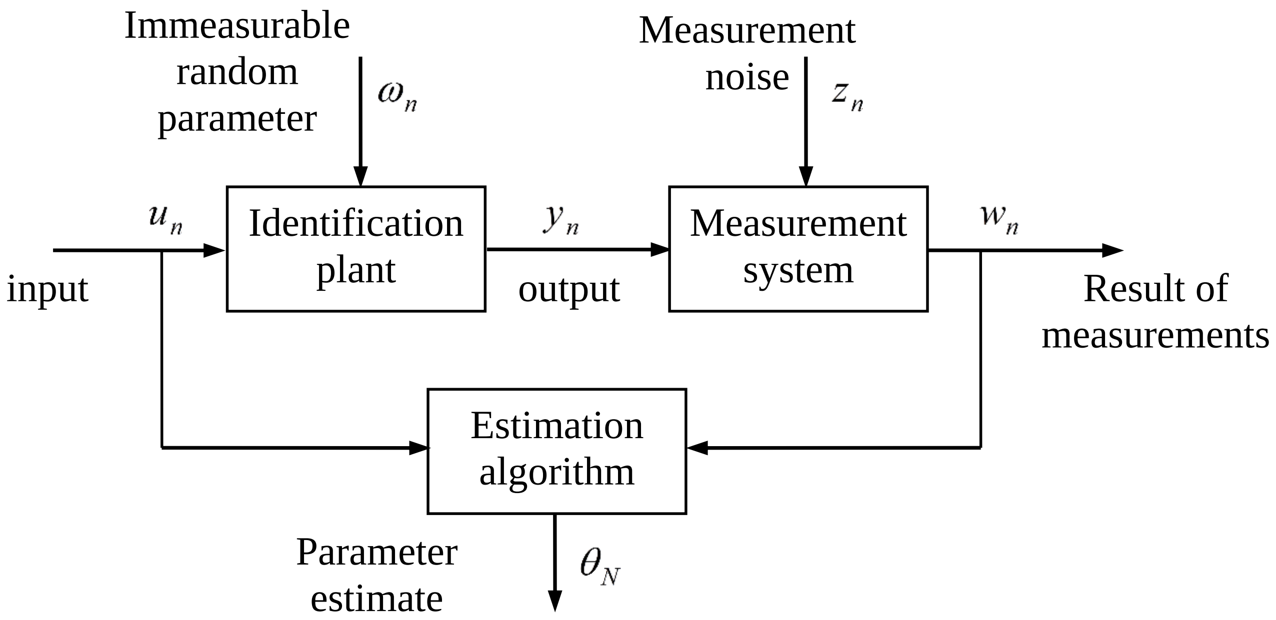
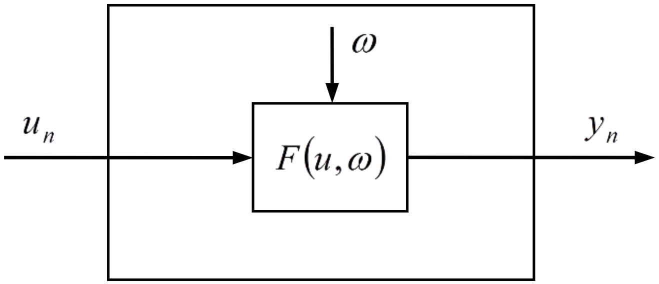
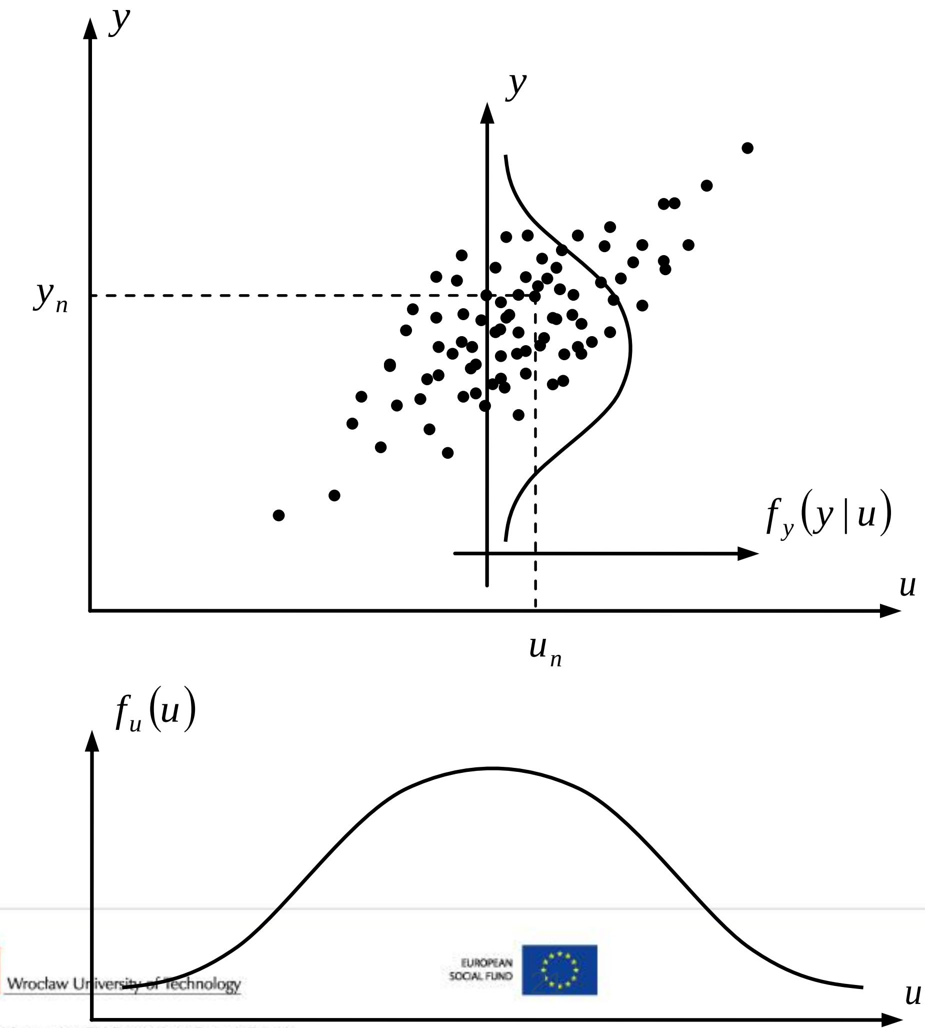
Plant parameter estimation problem





$$(u_n, y_n), n = 1, 2, \ldots, N$$

are values of random variables $(\underline{u}, \underline{y})$



Choice of the best model, probabilistic case

Two possible cases

Full a'priori knowledge

- joint probability density function $f(u, y)$ of random variables $(\underline{u}, \underline{y})$

or

- conditional probability density function $f_y(y|u)$

and mrginal probability density function

$$f_u(u)$$

are known

Incomplete probabilistic information

joint probability density function of random variables $(\underline{u}, \underline{y})$ exist, but is not known.

Measurements:

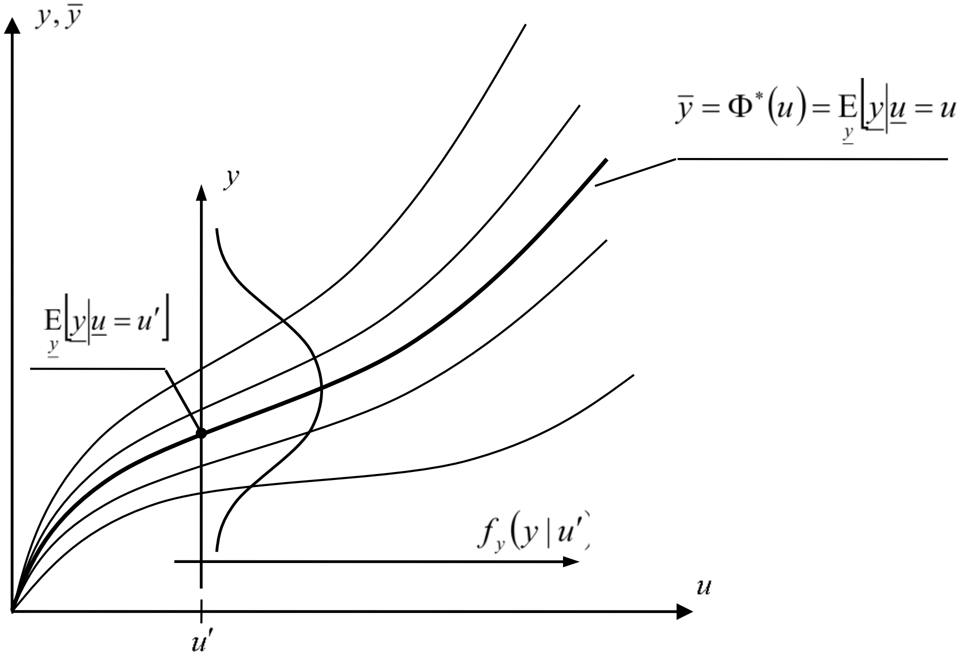
$$(u_n, y_n),\ n = 1, 2, \ldots, N$$

are values of $(\underline{u}, \underline{y})$

Full a’priori knowledge

- Regression of the I type

$$\bar{y} = \Phi^{*}(u) = \underset{\underline{y}}{E}\left[\underline{y}\middle|\underline{u} = u\right] = \int_{\mathcal{Y}} y \; f\left(y\middle|u\right) dy$$



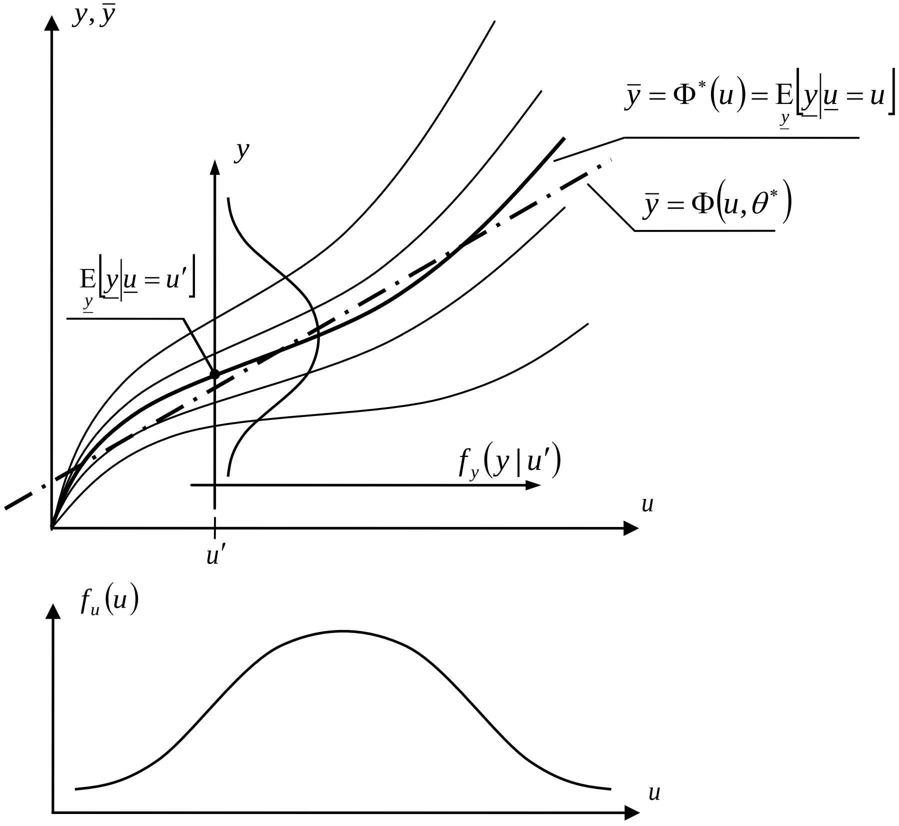
Full a’priori knowledge

- Regression of the II type

$$q(y,\bar{y}) = [y-\bar{y}]^T [y-\bar{y}]$$

$$Q(\theta) = \int_{\mathcal{U}} \int_{\mathcal{Y}} [y-\Phi(u,\theta)]^T [y-\Phi(u,\theta)] \times$$

$$\times\, f(u,y)\, dy\, du$$



Full a'priori knowledge

$$Q(\theta)=\int_{\mathscr{U}}\int_{\mathscr{Y}}\left(y-\Phi^{*}(u)\right)^{2} f(u,y)\,dy\,du+\int_{\mathscr{U}}\left(\Phi^{*}(u)-\Phi(u,\theta)\right)^{2} f_{u}(u)\,du$$

$$\theta^{*}\rightarrow\min_{\theta} Q(\theta)=\min_{\theta}\int_{\mathscr{U}}\left(\Phi^{*}(u)-\Phi(u,\theta)\right)^{2} f_{u}(u)\,du$$

the I type regression   weight function

The II type regression is the best approximation of the I type regression.

Unknown a Priori Knowledge

Empirical Estimation of the Performance Index

Empirical Estimation of the Performance Index

Empirical Probability Density Functions

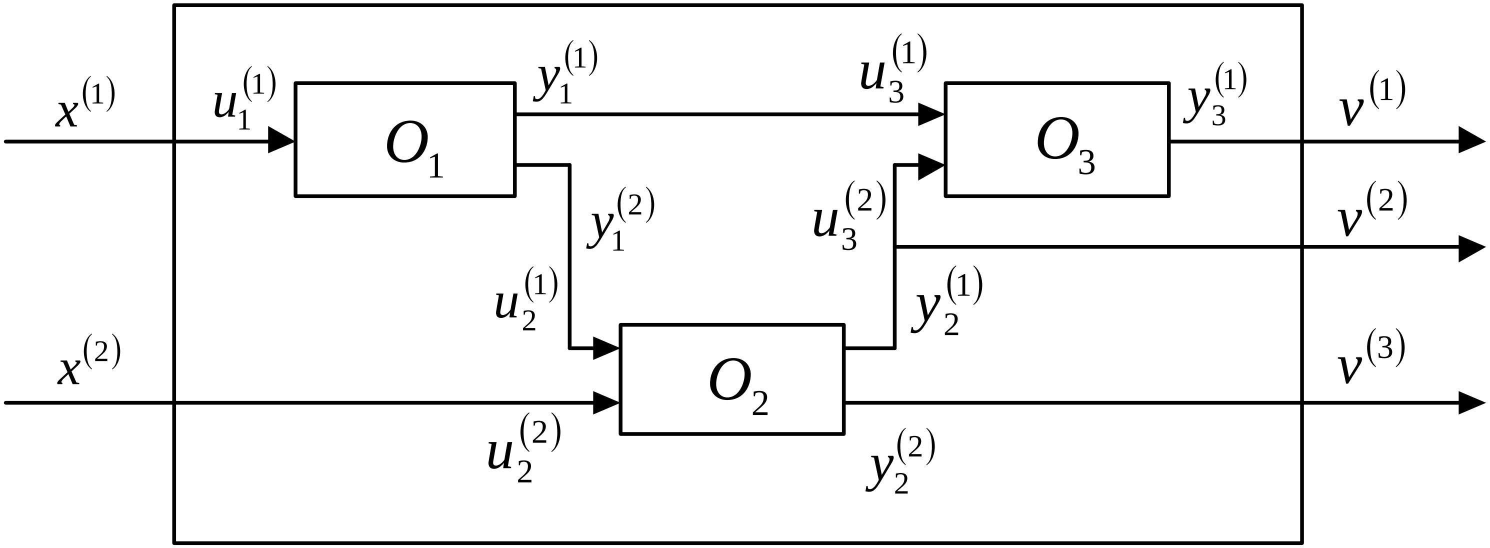
Unknown parameters of the probability density functions

Empirical Probability Density Functions

Non parametric – Parzen estimation

⋮

Complex systems description



Example of complex system

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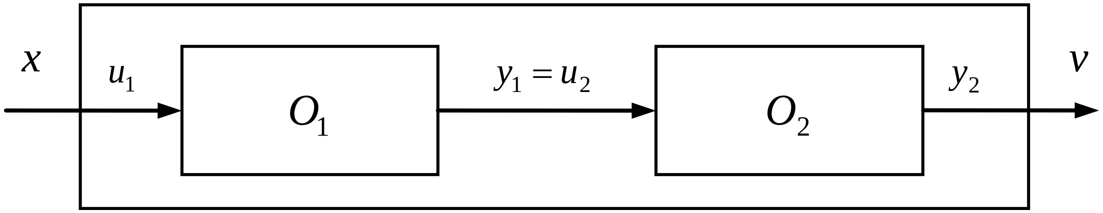
Complex systems identification problems

- Identification with restricted measurements posibilities
- Local and global identification
- Multistage identification
- Compleks of operation systems

Identification of complex systems with restricted measurement possibilities

The following examples show the problem.



Cascade structure of two elements

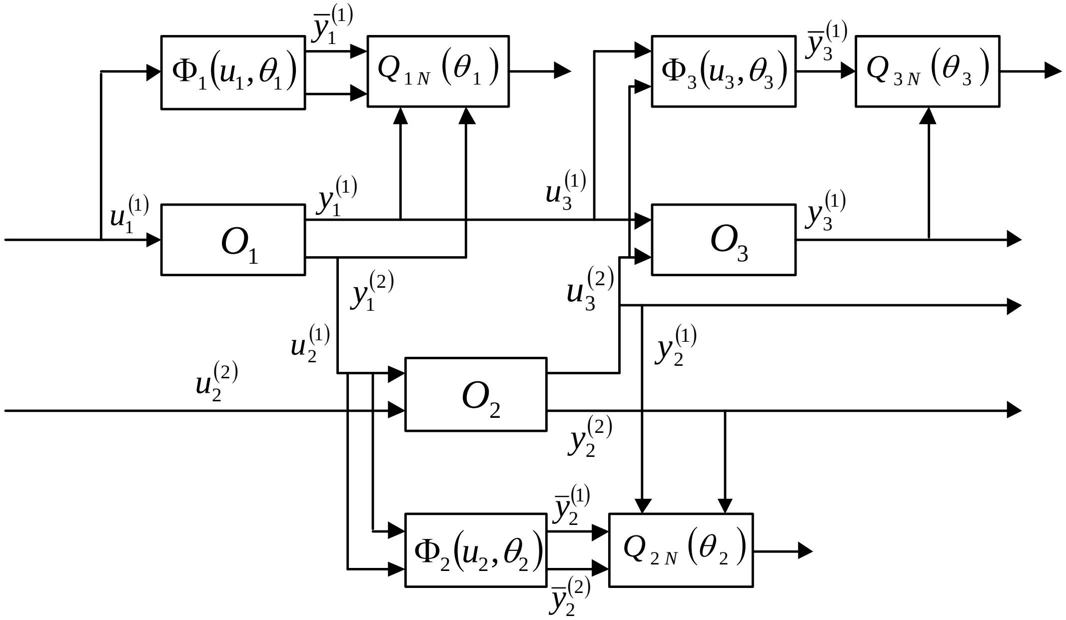
For the above case the system description has the form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x = \begin{bmatrix} x \\ y_2 \end{bmatrix},$$

$$v = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_2 .$$

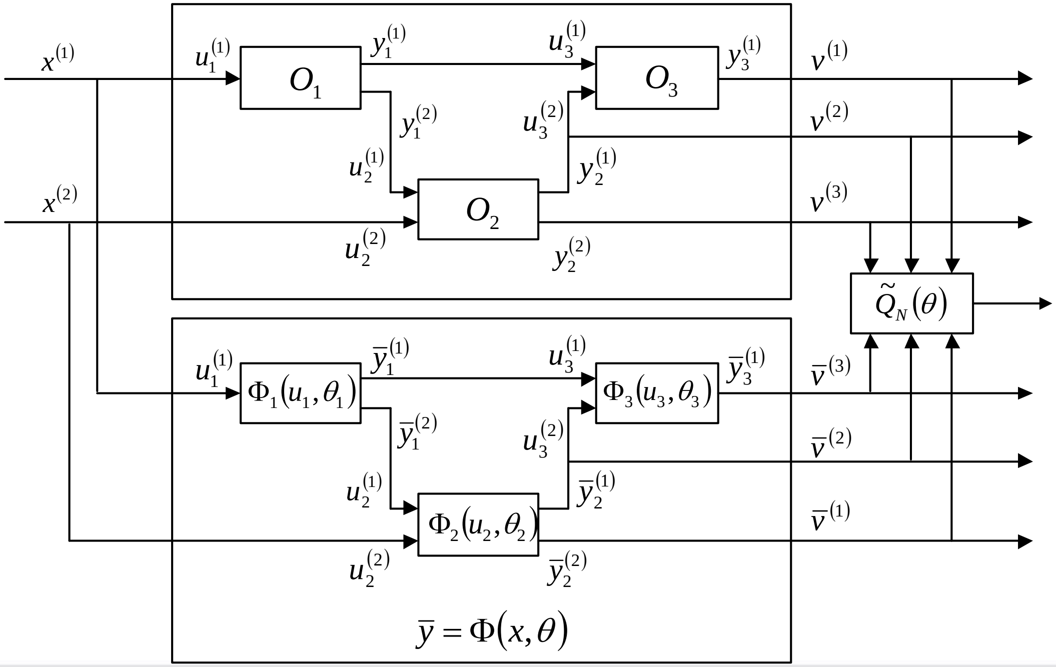
Choice of the best model of complex system

- Locally optimal model of complex system



Choice of the best model of complex system

- Globally optimal model of complex system



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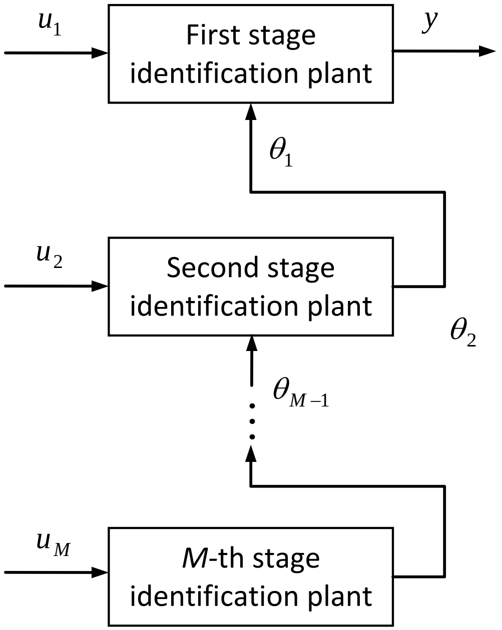
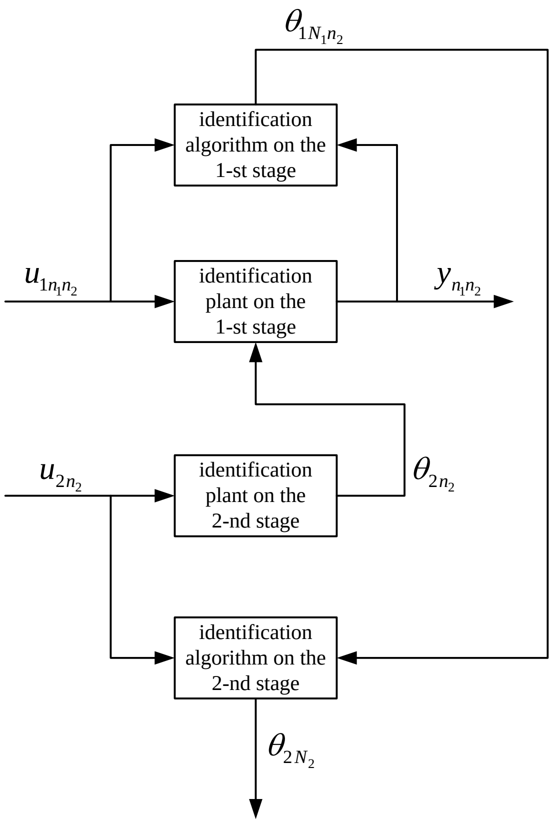
Two stage identification and it’s applications

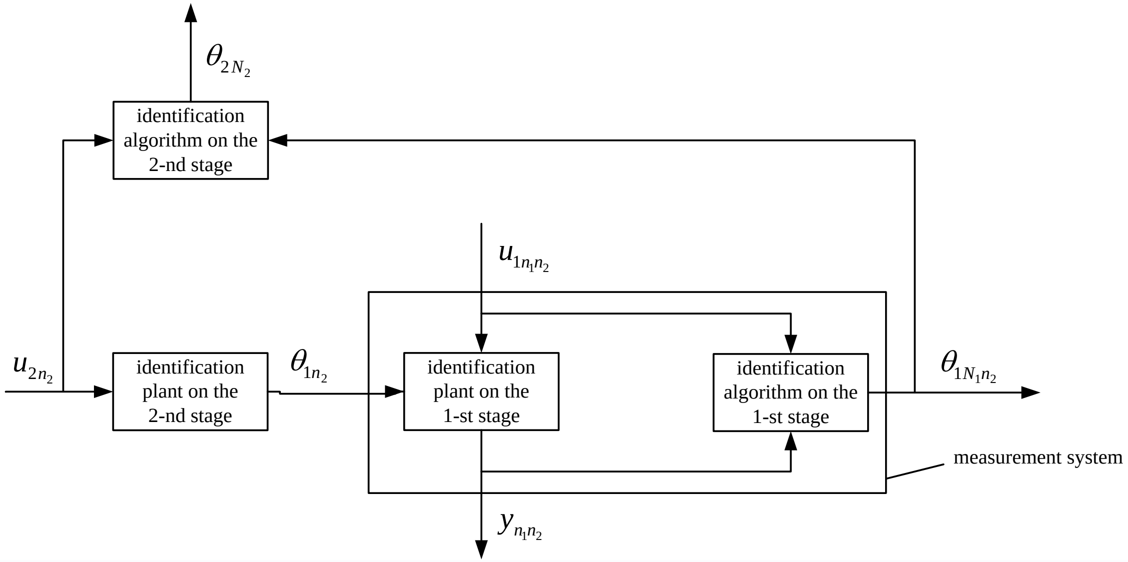


Two stage identification and it's applications



Two stage identification

- Space decomposition
- Time decomposition



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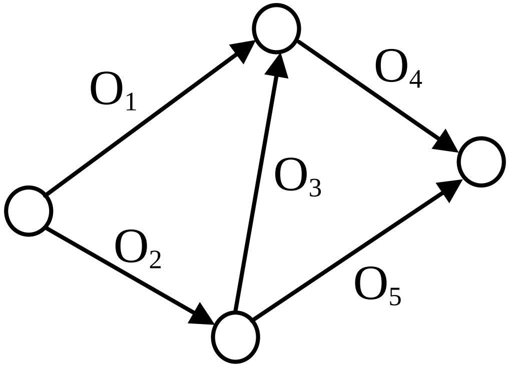
Identification of complex of operations



$$T_m = F_m(u_m, a_m), \quad m = 1,2,\ldots,M, \qquad T = H(T_1, T_2, \ldots, T_M)$$

H – function determining the total runtime of complex of operation

$F_1, F_2, \ldots, F_M$ – known functions

$a_1, a_2, \ldots, a_M$ – unknown parameters

Basic optimization task formulation

Decision variables: $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix}$

Objective function: $y = F(x)$

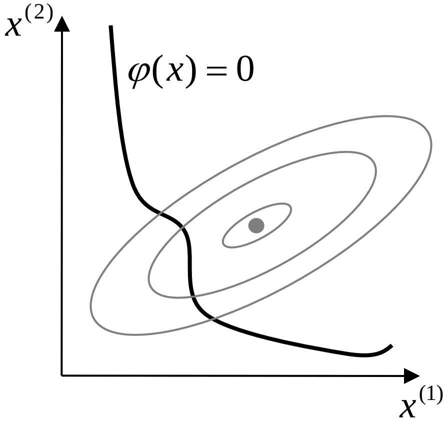
Set of feasible decisions (commonly defined by variables domain and constraints):

$$x \in \mathscr{D}_x$$

Optimization task: $x^* \to F(x^*) = \min\limits_{x^* \in \mathscr{D}_x} F(x),$ x^* – optimal decision

$$\min F(x) = -\max\bigl(-F(x)\bigr)$$

General classification of optimization tasks



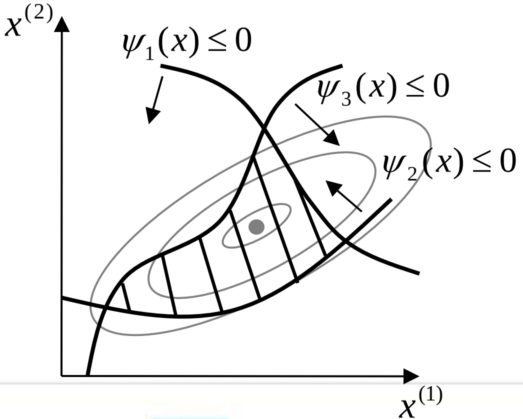
Unconstrained optimization: $\mathscr{D}_x = \mathscr{R}^S$

Optimization under equality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi_1(x)=0, \varphi_2(x)=0, \ldots, \varphi_L(x)=0, L \le S \right\}$$

Optimization under inequality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$



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Analytical methods

- Unconstrained optimization
- Lagrange multipliers method – equality constraints
- Kuhn-Tucker conditions – inequality constraints

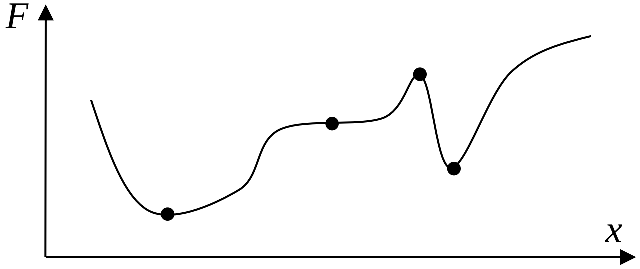
Unconstrained optimization

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x)$

Assumption: $F(x)$ is continuous and differentiable.

Necessary condition for x^* to be local minima: $\nabla_x F(x^*) = 0_S$

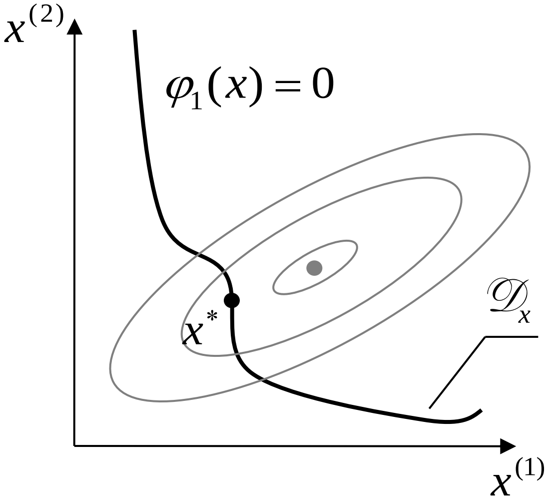
If $F(x)$ is convex function, then above equation is sufficient condition for x^* to be global minima.



Optimization under equality constraints

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \ \varphi_1(x) = 0, \varphi_2(x) = 0, \ldots, \varphi_L(x) = 0, \ L \leq S \right\}$$



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Optimization under equality constraints

- The method of Lagrange multipliers

Lagrange function:

$$L(x,\lambda) = F(x) + \sum_{l=1}^{L} \lambda_l \varphi_l(x) = F(x) + \lambda^T \varphi(x)$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_L \end{bmatrix}, \quad \varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_L(x) \end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x,\lambda)\Big|_{x^*,\lambda^*} = 0_S$$

$$\nabla_\lambda L(x,\lambda)\Big|_{x^*,\lambda^*} = 0_L$$

If and only if

$$\operatorname{rank} G(x) = \operatorname{rank} \left[G(x) \;\vdots\; -\nabla_x F(x) \right],$$

Where: $G(x) = \left[\nabla_x \varphi_1(x) \;\vdots\; \nabla_x \varphi_2(x) \;\vdots\; \cdots \;\vdots\; \nabla_x \varphi_L(x) \right]$

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Optimization under equality constraints

- The generalized method of Lagrange multipliers

Generalized Lagrange function:

$$L(x,\lambda,\lambda_0)=\lambda_0 F(x)+\sum_{l=1}^{L}\lambda_l\varphi_l(x)$$

Necessary conditions of optimality:

$$\nabla_x L(x,\lambda,\lambda_0)\Big|_{x^*,\lambda^*,\lambda_0}=0_S$$

$$\nabla_\lambda L(x,\lambda,\lambda_0)\Big|_{x^*,\lambda^*,\lambda_0}=0_L$$

Optimization under equality constraints

- The generalized method of Lagrange multipliers

$$\nabla_x L(x,\lambda,\lambda_0) = \lambda_0 \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$$

1° $\quad \lambda_0 \neq 0 \quad \nabla_x F(x) + \sum_{l=1}^{L} \frac{\lambda_l}{\lambda_0} \nabla_x \varphi_l(x) = 0_S \Rightarrow \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l' \nabla_x \varphi_l(x) = 0_S$

$\lambda_0 = 1 \quad \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$ We obtain regular solutions.

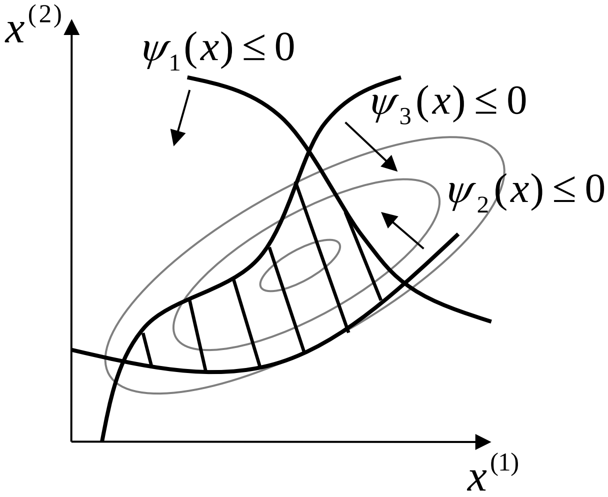
2° $\quad \lambda_0 = 0 \quad \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$ We obtain irregular solutions.

Second order condition of optimality requires analysis of $H(x,\lambda,\lambda_0) = \nabla_{xx}^2 L(x,\lambda,\lambda_0)$.

Optimization under inequality constraints

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$



Optimization under inequality constraints

Kuhn-Tucker conditions

Lagrange function:

$$L(x,\mu)=F(x)+\mu^T\psi(x) \quad \Leftrightarrow \quad L(x,\mu)=F(x)+\sum_{m=1}^{M}\mu_m\psi_m(x)$$

$$\mu=\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_M\end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*}=0_S$$

$$\mu^T\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*}=0$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*}\le 0_M$$

$$\mu^*\ge 0_M$$

If solution is regular

$$\alpha=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_S\end{bmatrix} \quad \beta=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_S\end{bmatrix}$$

$$\alpha\le\beta\Rightarrow\forall_{s=1,\ldots,S}\,\alpha_s\le\beta_s$$

Optimization under inequality constraints Kuhn – Tucker rolls

Regularity Conditions

1. Karlin: constraints $\psi_1(x), \psi_2(x), \ldots, \psi_M(x)$ - linear

2. Slater: constraints $\psi_1(x), \psi_2(x), \ldots, \psi_M(x)$ - convex functions and feasible set is not empty

3.Fiacco – Mac Cormica: in the optimal point gradients of all active constraints are linear independent, i.e.:

$$\forall\, m \in I\left(x^*\right) \quad \nabla_x \psi_m\left(x^*\right)\Big|_{x=x^*}$$ are linear independent

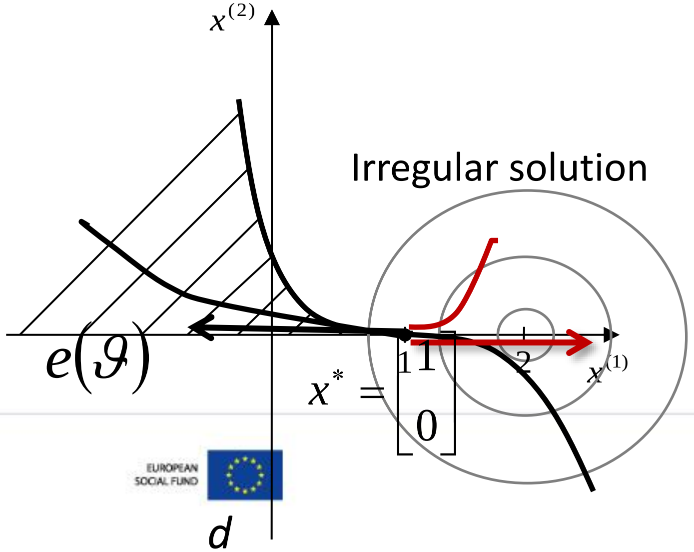
4. Zangwil: $\mathscr{D}(x^*) = \overline{D}(x^*)$

5. Kuhna – Tucker’a: for each direction $d \in \mathscr{D}(x^*)$ there exists regular curve starting in the point x^* tangent to that direction

$$\forall\, d \in \mathscr{D}(x^*) \quad \exists\, e(\vartheta), \quad \vartheta \in [0,1]$$

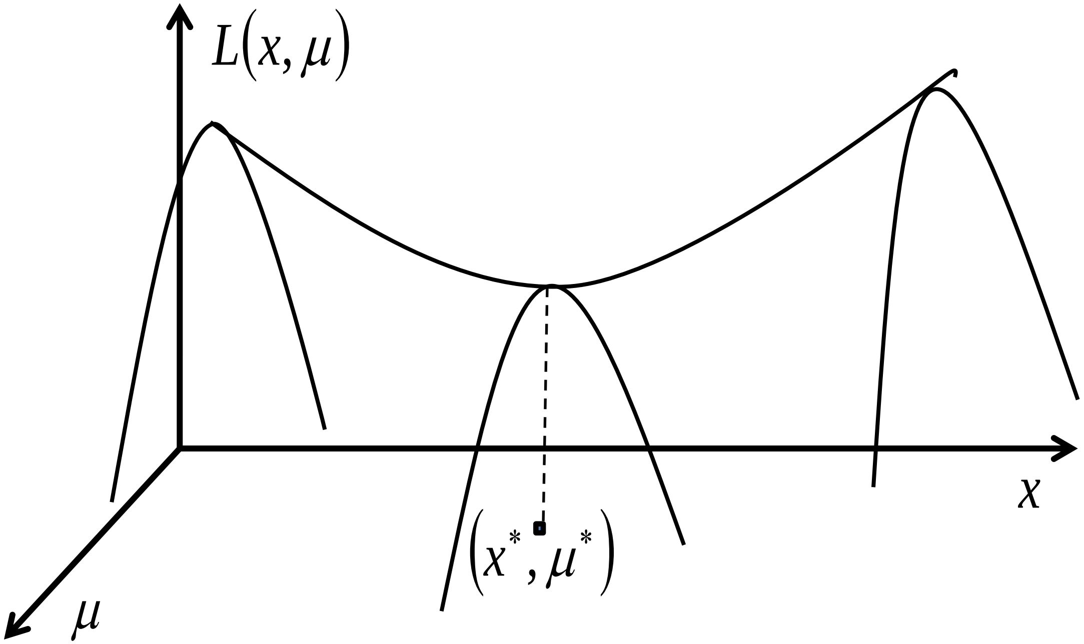
$$e(\vartheta) = \begin{bmatrix} e_1(\vartheta) \\ e_2(\vartheta) \\ \vdots \\ e_S(\vartheta) \end{bmatrix}$$

- $e(0) = x^*$
- $e(\vartheta) \in D_x \quad \forall\, \vartheta \in [0,1]$
- $\dfrac{de(\vartheta)}{d\vartheta}\Big|_{\vartheta=0} = \tau \cdot d$



Saddle point

Saddle point $\left(x^*,\mu^*\right)$



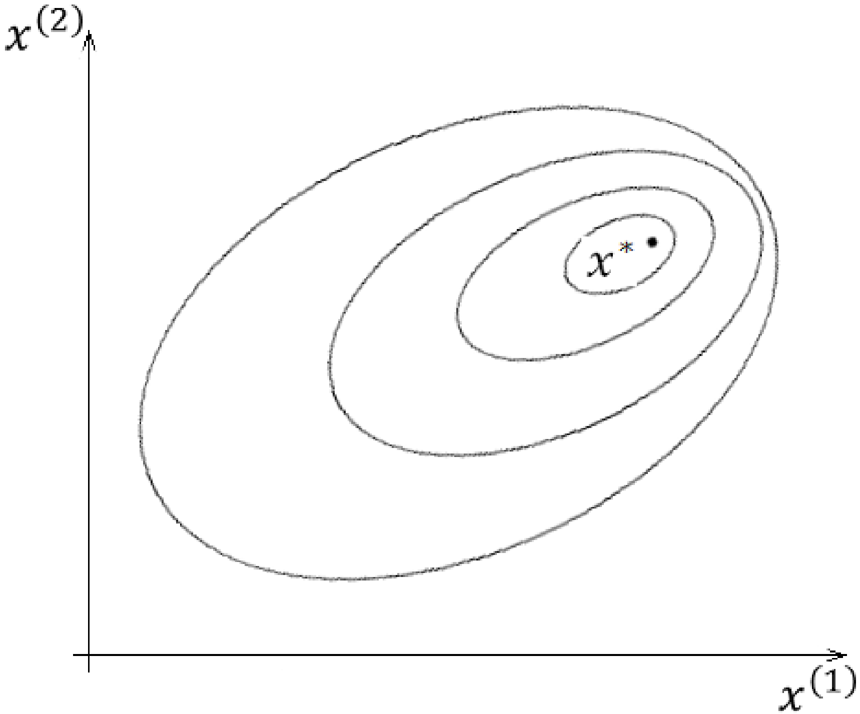
$$L\left(x^*,\mu^*\right)\le L\left(x,\mu^*\right) \quad \forall x\in\mathscr{D}(x)\subseteq\mathscr{R}^S$$

$$L\left(x^*,\mu\right)\le L\left(x^*,\mu^*\right) \quad \forall\mu\ge 0_M$$

$$L\left(x^*,\mu^*\right)=\min_{x\in\mathscr{D}(x)}\max_{\mu\ge 0_M} L\left(x,\mu\right)$$

Numerical optimization methods

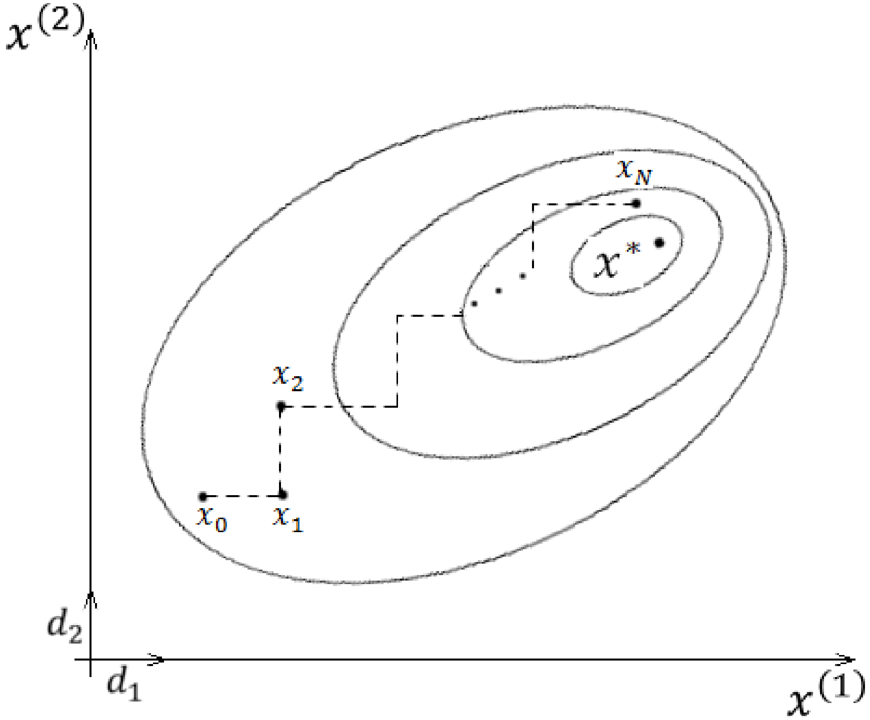
$$x^* \to F(x^*) = \min_{x \in D_x} F(x)$$



Analytical methods has drawbacks, when:

1. The goal function F and constraints φ, ψ are nonlinear.
2. Functions F, φ and ψ are non-differentiable
3. Mathematical formula describing functions F, φ and ψ is not available, it can only be „measured”
4. Large dimension of decision variables vector

Numerical optimization methods

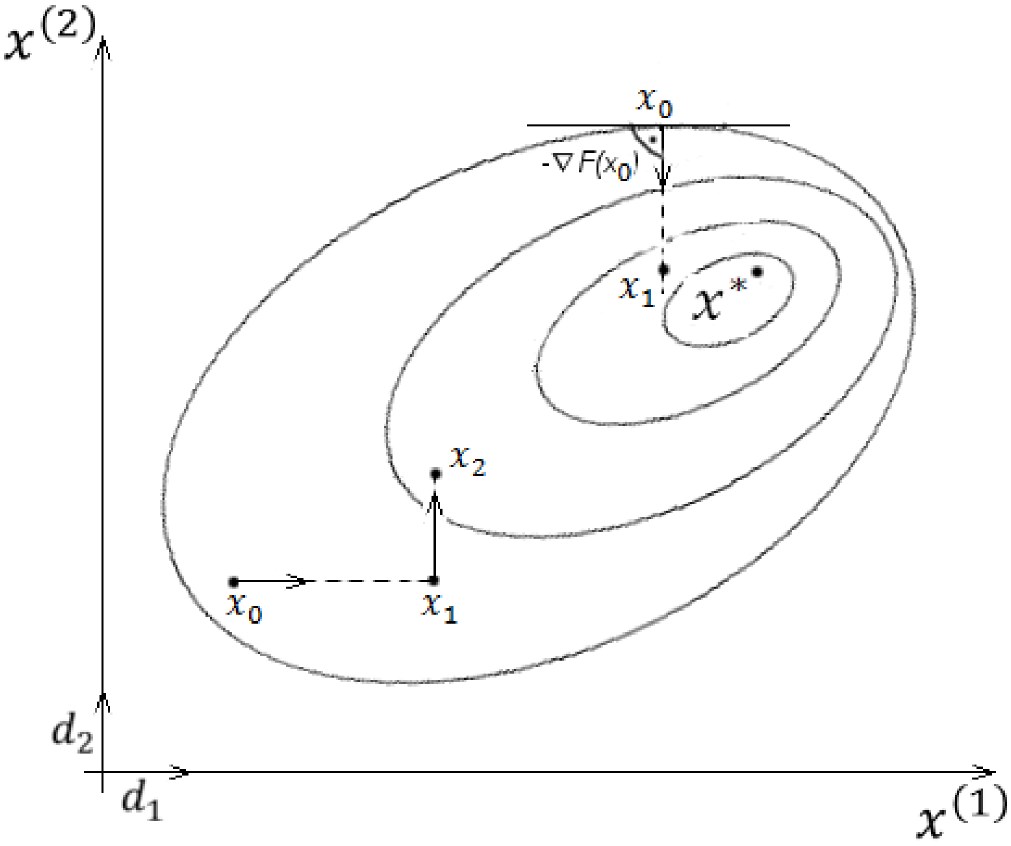


Algorithm
$x_{n+1} = \Psi(x_n), x_0$

- Choice of the search direction.
- Line search optimization.
- Stopping conditions.

$$x_0, x_1, \dots, x_n, \dots, x_N \approx x^*$$
$$F(x_0) > F(x_1) > \ \dots > F(x_n) > \ \dots > F(x_N) \approx F(x^*)$$

Choice of the search direction



- Basis of search directions – non-gradient methods.
- Search directions based on gradient vectors – gradient-based methods.

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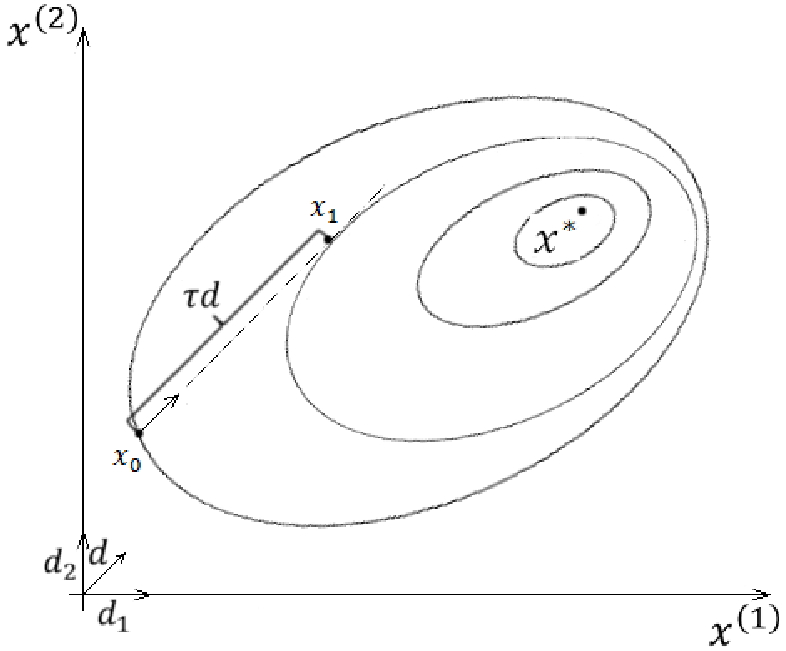
Line search optimization



x_0 – initial solution
x_1 – next solution
d – search direction
τ – step size

$$\tau^* \to F(x_0 + \tau^* d) = \min_{\tau} F(x_0 + \tau d)$$

x_0, d – fixed

$$F(x_0 + \tau d) \triangleq f(\tau)$$

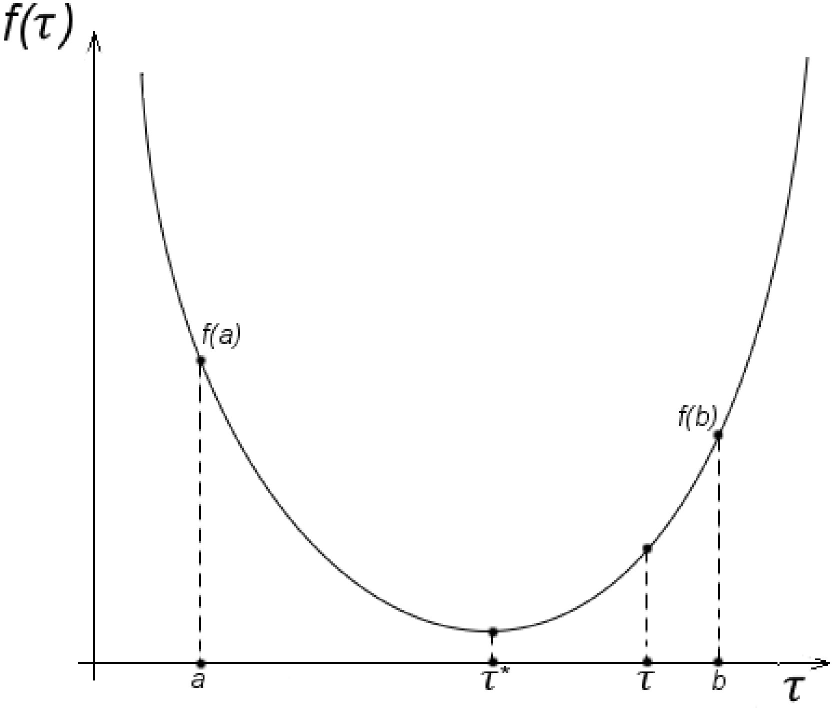
$f(\tau)$ – a single variable function
(of the step size τ)

$$\tau^* \to f(\tau^*) = \min_{\tau} f(\tau)$$

line search optimization ≡ optimization of a single variable function

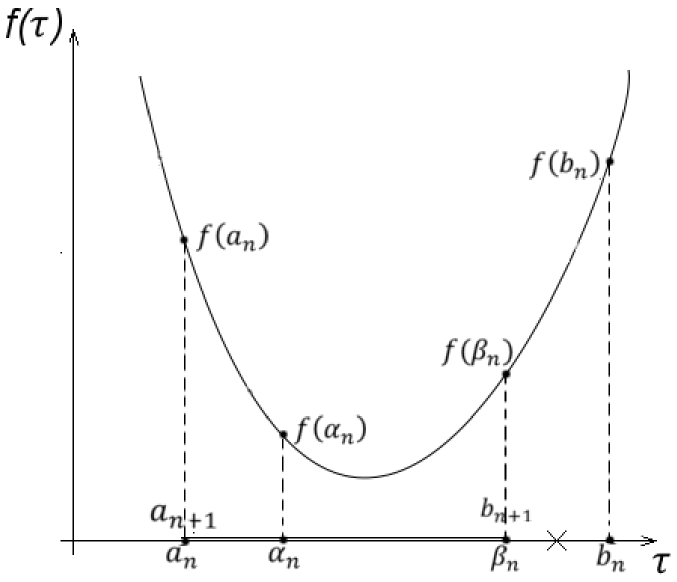
Reducing the interval of uncertainty

Assumption: $\tau^* \in [a, b]$

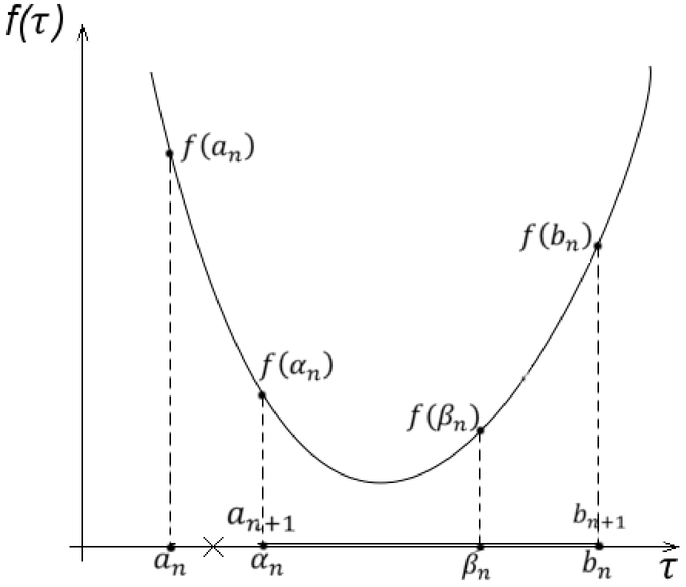


Splitting the section into two parts

$$f(\alpha_n) ? f(\beta_n)$$

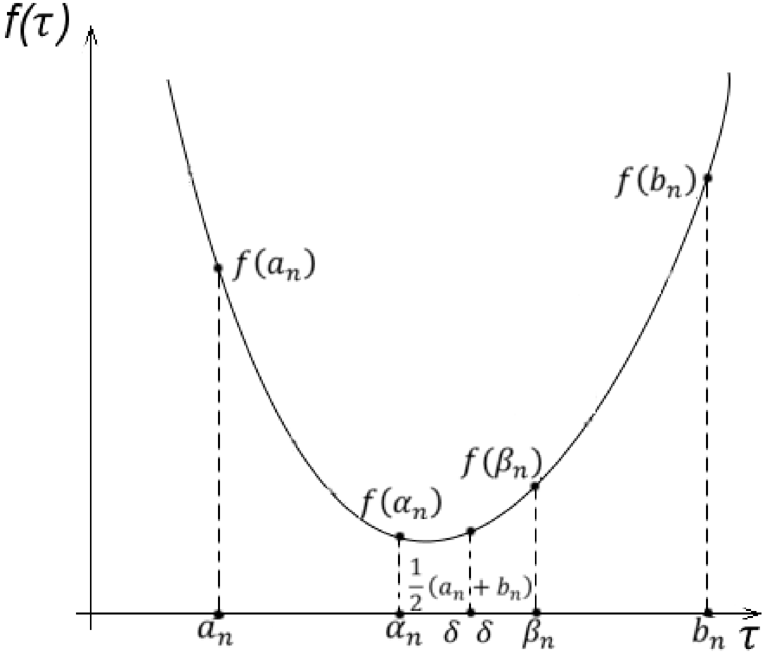


$$f(\alpha_n) \leq f(\beta_n)$$
$$a_{n+1} := a_n$$
$$b_{n+1} := \beta_n$$



$$f(\alpha_n) > f(\beta_n)$$
$$a_{n+1} := \alpha_n$$
$$b_{n+1} := b_n$$

Dichotomous search method



$$\alpha_n = \frac{1}{2}(a_n + b_n) - \delta$$

$$\beta_n = \frac{1}{2}(a_n + b_n) + \delta \qquad N = ?$$

Input data: $a_0, b_0, \varepsilon, \delta$

Step 0: $n = 0$

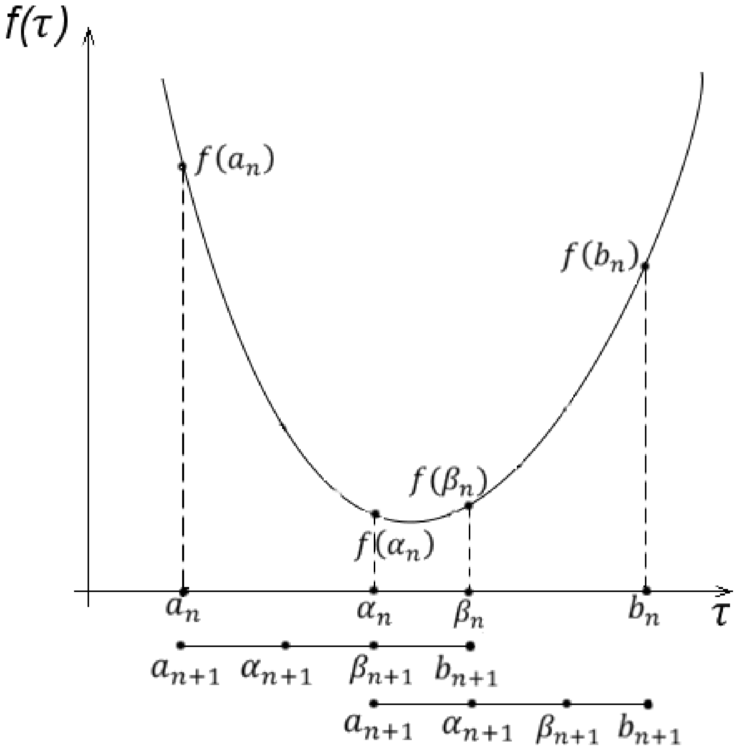
Step 1: $\alpha_n = \frac{1}{2}(a_n + b_n) - \delta$

$\beta_n = \frac{1}{2}(a_n + b_n) + \delta$

Step 2: If $f(\alpha_n) \leq f(\beta_n)$ then
$a_{n+1} := a_n, b_{n+1} := \beta_n,$
otherwise
$a_{n+1} := \alpha_n, b_{n+1} := b_n.$

Step 3: If $|b_{n+1} - a_{n+1}| \geq \varepsilon$ then
$n := n + 1$, go to 1,
otherwise
$\tilde{\tau} = \frac{1}{2}(a_{n+1} + b_{n+1})$ (STOP)

The golden section method

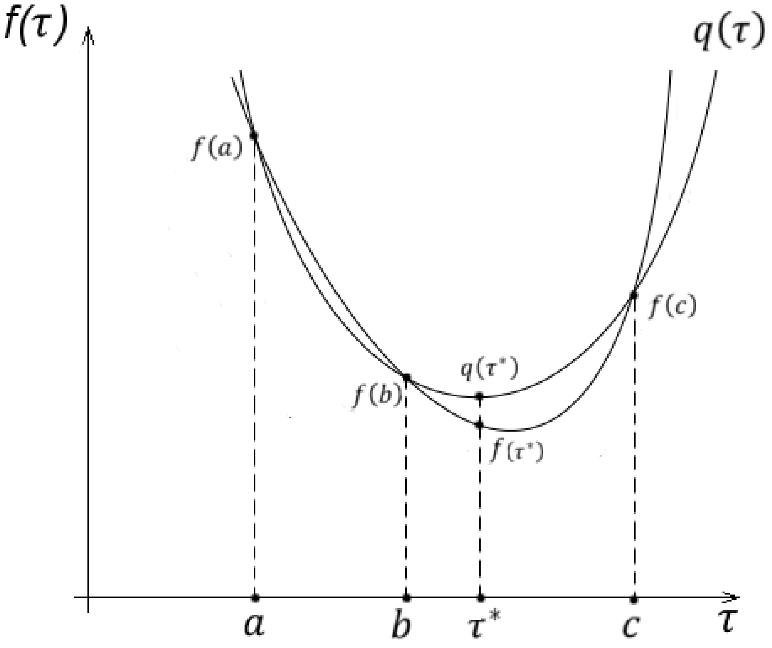


$$\gamma^2 + \gamma \; - 1 = 0$$

$$\gamma = \frac{\sqrt{5}-1}{2} \approx 0.618 \qquad N = ?$$

```
Input data: $a_0, b_0, \varepsilon, \gamma = \frac{\sqrt{5}-1}{2}$
Step 0:   $n = 0$
          $\alpha_0 = b_0 + \gamma(a_0 - b_0)$
          $\beta_0 = a_0 + \gamma(b_0 - a_0)$
Step 1:   If $|b_n - a_n| < \varepsilon$, then
          $\tilde{\tau} = \frac{1}{2}(a_n + b_n)$(STOP)
          otherwise go to 2
Step 2:   If $f(\alpha_n) \le f(\beta_n)$ then
           $a_{n+1} := a_n, b_{n+1} := \beta_n,$
           $\beta_{n+1} := \alpha_n, \quad \alpha_{n+1} := \beta_n + \gamma(a_n - b_n)$
           $n := n + 1$, go to 1
           otherwise
           $a_{n+1} := \alpha_n, b_{n+1} := b_n,$
           $\alpha_{n+1} := \beta_n, \quad \beta_{n+1} := \alpha_n + \gamma(b_n - \alpha_n)$
           $n := n + 1$, go to 1
```

Quadratic-fit line search method



$$a < b < c$$
$$f(a) \geq f(b)$$
$$f(b) \leq f(c)$$

$q(\tau)$ – quadratic-fit function
τ^* - minimum of the function $q(\tau)$

$$q(\tau) = \frac{f(a)(\tau - b)(\tau - c)}{(a - b)(a - c)} + \frac{f(b)(\tau - a)(\tau - c)}{(b - a)(b - c)} + $$
$$+ \frac{f(c)(\tau - a)(\tau - b)}{(c - a)(b - c)}$$

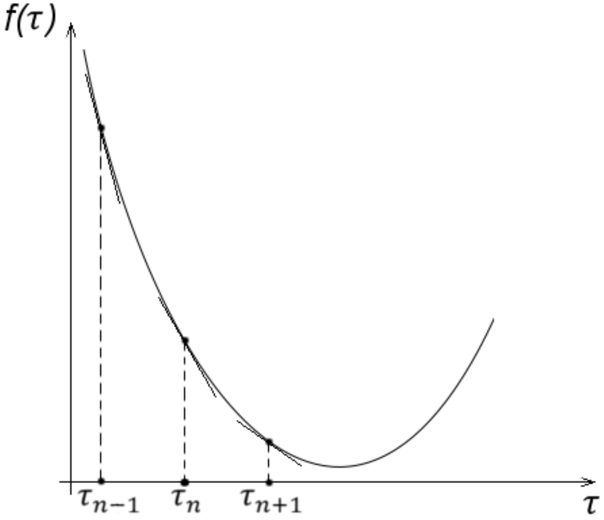
$$\tau^* = \frac{1}{2}\frac{f(a)\left(b^2 - c^2\right) + f(b)\left(c^2 - a^2\right) + f(c)(a^2 - b^2)}{f(a)(b - c) + f(b)(c - a) + f(c)(a - b)}$$

Line search using derivatives

$\tau_{n+1} = \tau_n - \gamma_n f'(t_n) \qquad \gamma_n > 0, \tau_0$

$\lim_{n\to\infty} \gamma_n = \gamma \qquad \sum_{n=0}^{\infty} \gamma_n = \infty$

e.g. $|\tau_{n+1} - \tau_n| < \varepsilon$ (STOP)



τ_0
$\tau_1 = \tau_0 - \gamma_0 f'(\tau_0)$
$\tau_2 = \tau_1 - \gamma_1 f'(\tau_1) = \tau_0 - \gamma_0 f'(\tau_0) - \gamma_1 f'(\tau_1)$
$\tau_{n+1} = \tau_n + \gamma_n f'(\tau_n) = \cdots = \tau_0 - \gamma_0 f'(\tau_0) - \gamma_1 f'(\tau_1) - \cdots - \gamma_n f'(\tau_n)$

$$|\tau_{n+1} - \tau_0| = |\sum_{k=0}^{n} \gamma_k f'(\tau_k)| \le \sum_{k=0}^{n} \gamma_k |f'(\tau_k)| \le \max_{0<k<n} |f'(\tau_k)| \sum_{k=0}^{n} \gamma_k$$

$$|\tau_\infty - \tau_0| \le \sum_{k=0}^{\infty} \gamma_k = \infty$$

Line search using sign of derivatives

$\tau_{n+1} = \tau_n - \vartheta_n sign[f'(\tau_n)]$

$\gamma_n f'(\tau_n) = \gamma_n |f'(\tau_n)| * sign\ f'(\tau_n) = \vartheta_n\ sign[f'(\tau_n)]$, where $\vartheta_n = \gamma_n |f'(\tau_n)|$

$\vartheta_n > 0$

$\lim_{n\to\infty} \vartheta_n = 0$, because $\lim_{n\to\infty} |f'(\tau_n)| = 0$, $\lim_{n\to\infty} \gamma_n = \gamma$

$$\sum_{n=0}^{\infty} \vartheta_n = \infty \qquad \lim_{n\to\infty} \vartheta_n = \lim_{n\to\infty} \gamma_n |f'(\tau_n)| = 0$$

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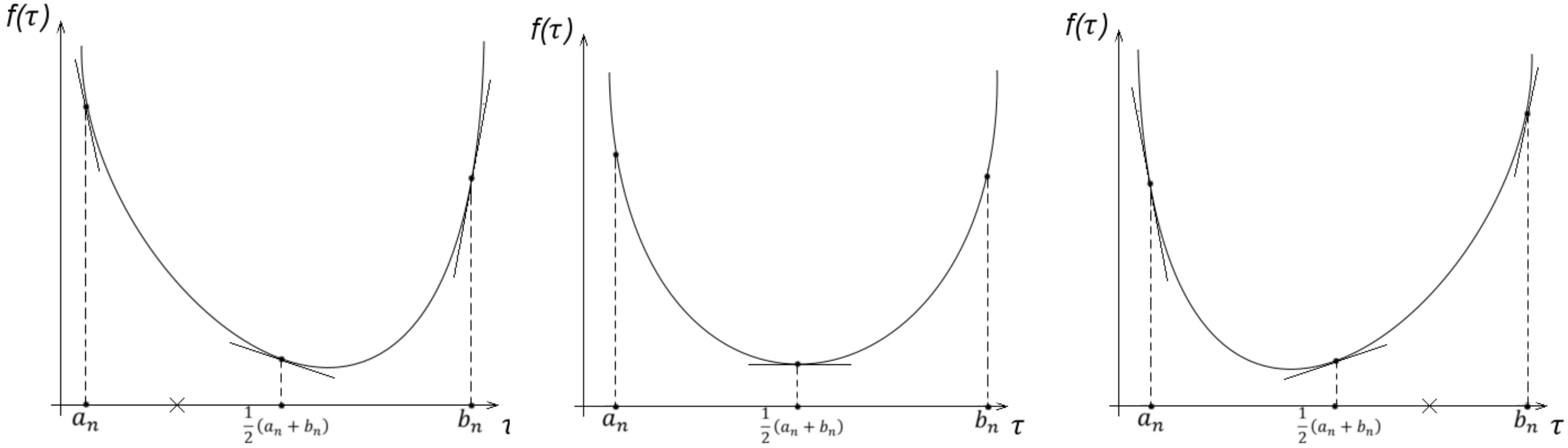
Bolzano method

$$sign\ a_n \neq sign\ b_n$$



$$sign\ f'(a_n) = signf'(\frac{1}{2}(a_n+b_n))$$

$$a_{n+1} \coloneqq \frac{1}{2}(a_n+b_n)$$

$$b_{n+1} \coloneqq b_n$$

$$f'\left(\frac{1}{2}(a_n+b_n)\right) = 0$$

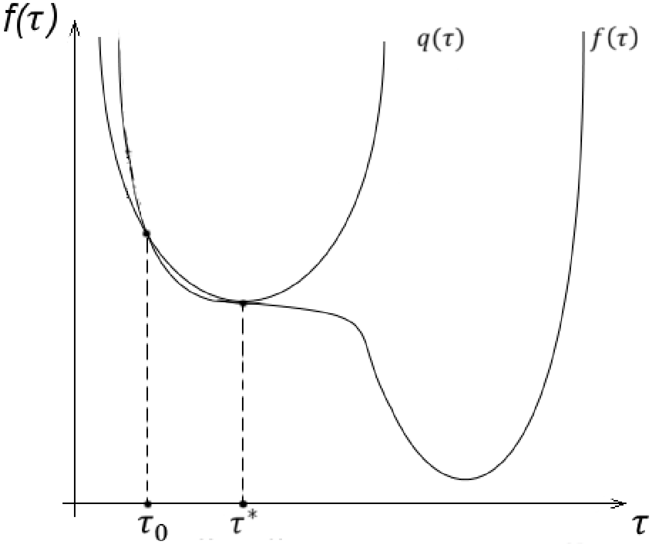
$$\tilde{\tau} \coloneqq \frac{1}{2}(a_n+b_n)$$

$$sign\ f'(b_n)$$
$$= signf'(\frac{1}{2}(a_n+b_n))$$

$$a_{n+1} \coloneqq a_n$$

$$b_{n+1} \coloneqq \frac{1}{2}(a_n+b_n)$$

Newton's method



τ_0

$$\tau_{n+1} = \tau_n - \frac{f'(\tau_n)}{f''(\tau_n)}$$

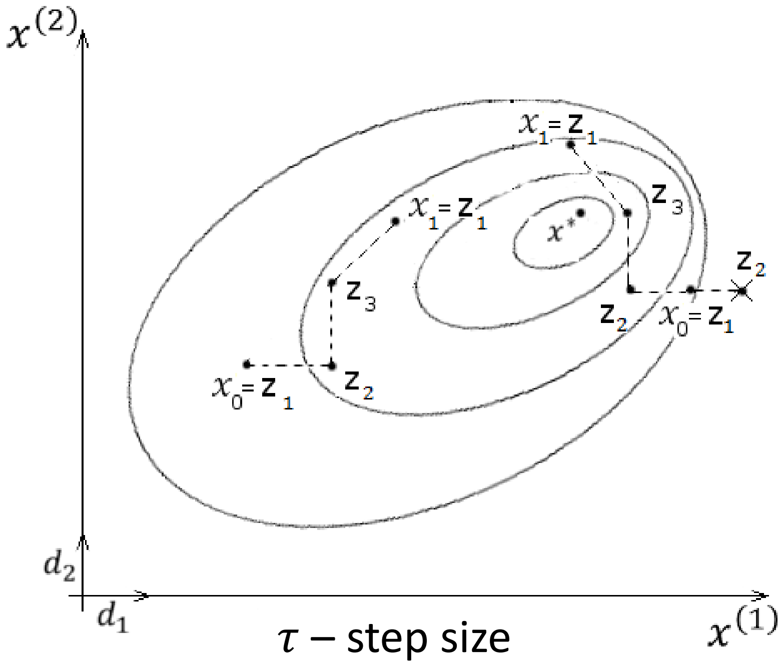
$|\tau_{n+1} - \tau_n| < \varepsilon$ (STOP)

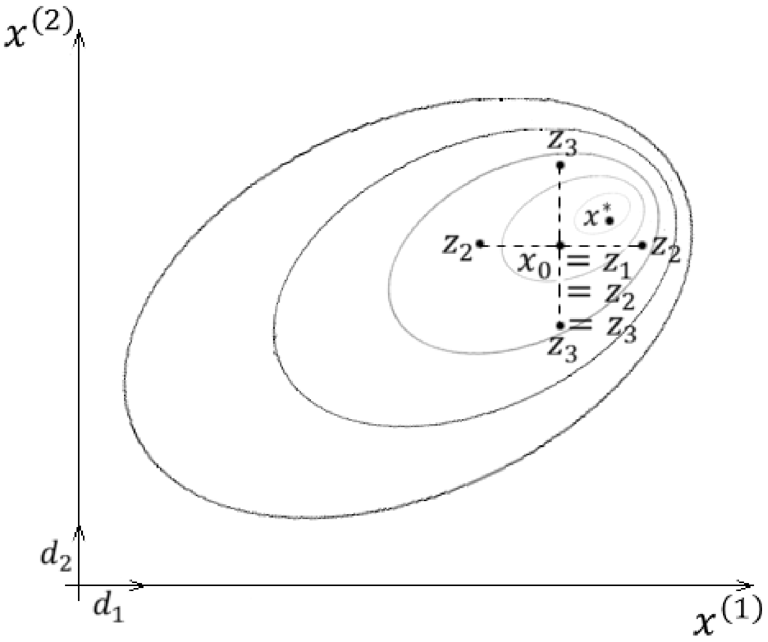
$$f(\tau) = \underbrace{f(\tau_0) + (\tau - \tau_0)f'(\tau_0) + \frac{1}{2}(\tau - \tau_0)^2 f''(\tau_0)}_{q(\tau)} + 0_3(|\tau - \tau_0|)$$

$$q'(\tau) = f'(\tau_0) + (\tau^* - \tau_0)f''(\tau_0) = 0$$

$$\tau^* = \tau_0 - \frac{f'(\tau_0)}{f''(\tau_0)}$$

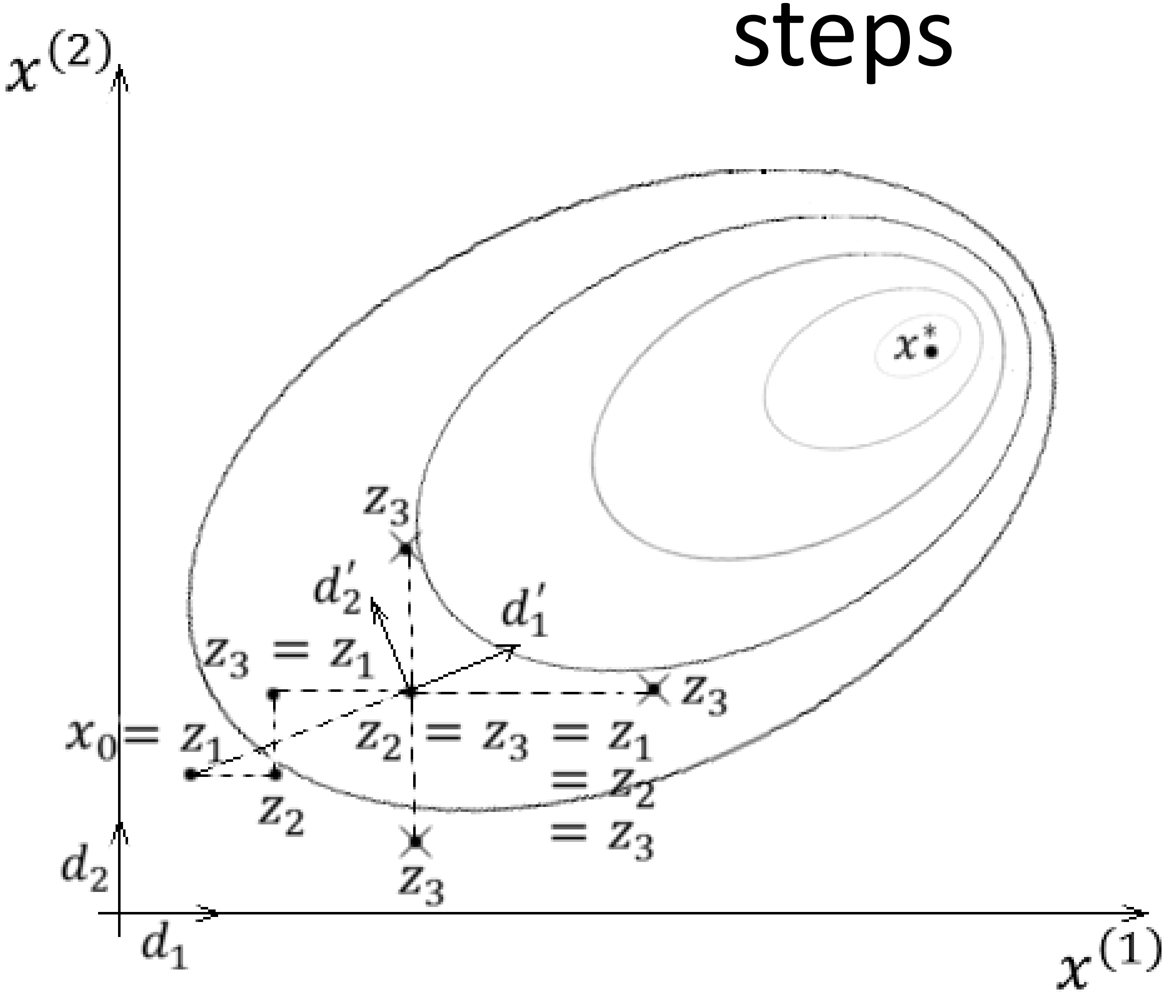
Method of Hooke and Jevees with discrete steps





τ – step size
$\alpha > 1$ exploratory step size
$\beta \in (0,1)$ acceleration factor
$\tau := \tau\beta$

Method of Rosenbrock with discrete steps



τ – step size

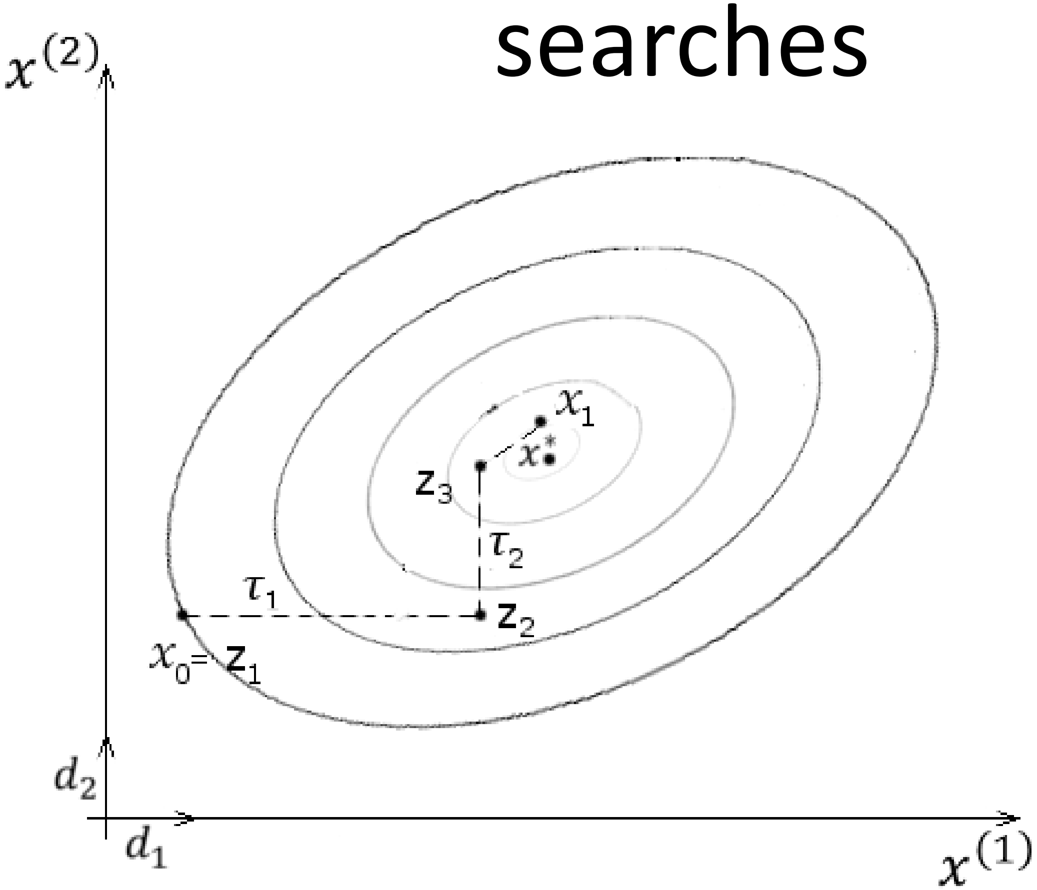
$\alpha > 1$ – exploratory step size acceleration

$\beta \in (-1, 0)$ – acceleration factor

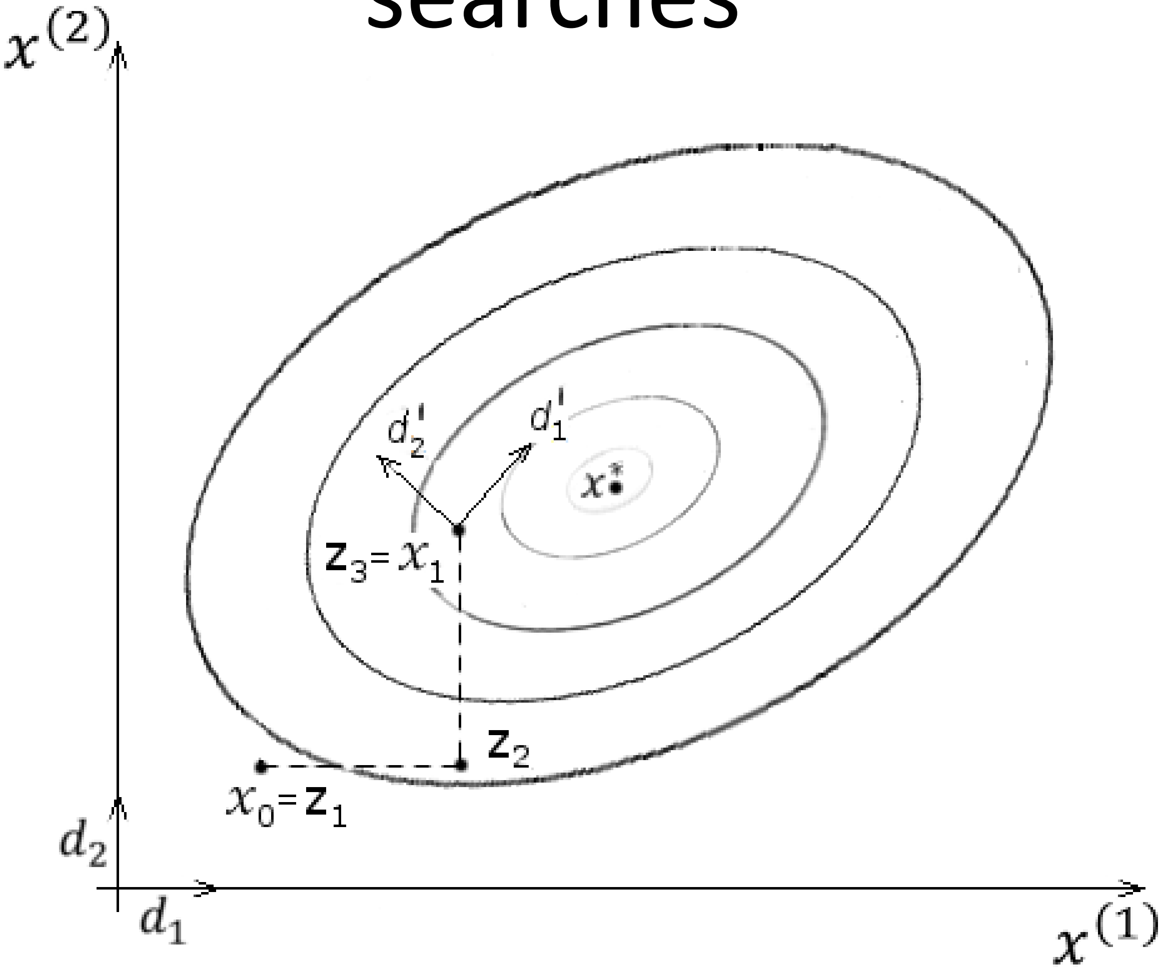
$\tau_s \coloneqq \tau_s \alpha$

$\tau_s \coloneqq \tau_s \beta$

Method of Hooke and Jeeves using line searches



Method of Rosenbrock using line searches

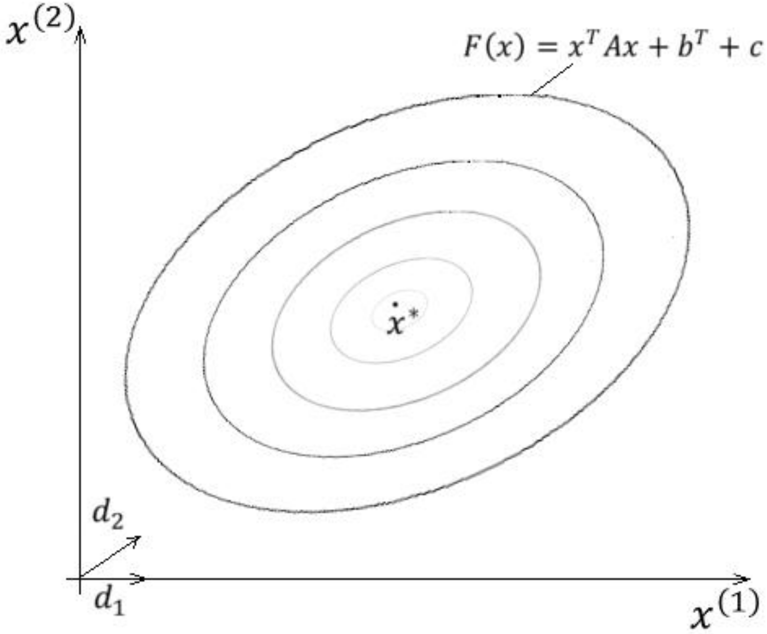


Powell’s method – conjugate directions

$d_1, d_2, \dots, d_S$ - conjugated directions,

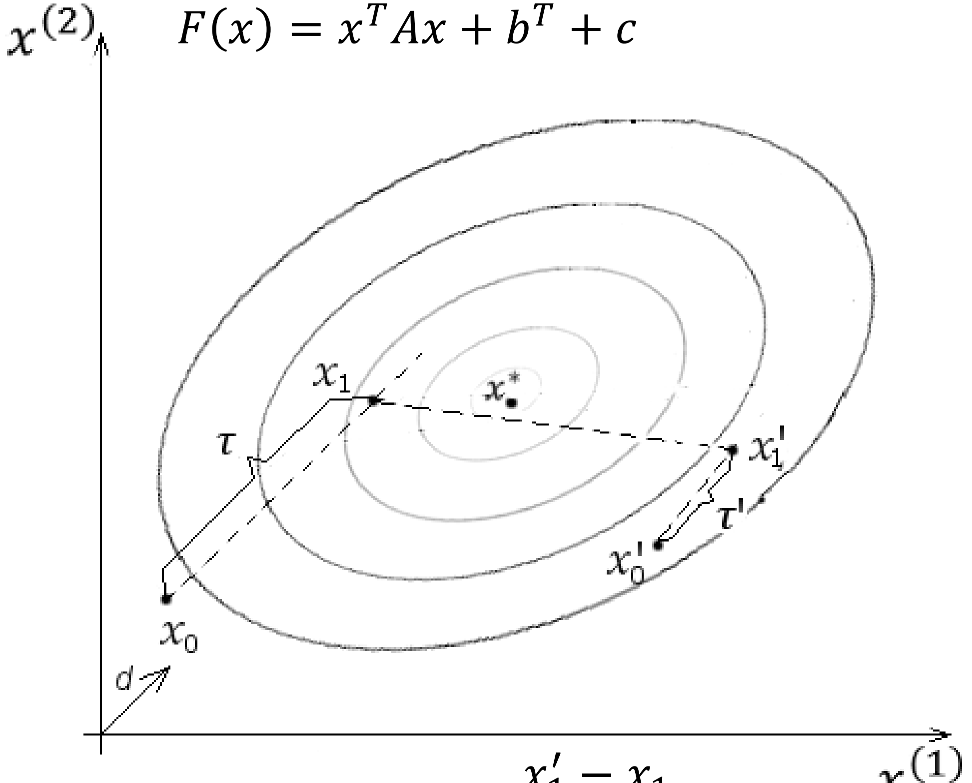
A – symmetric, positively defined matrix

$$d_i^T A d_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$



Powell's method – conjugate directions

$$F(x) = x^T Ax + b^T + c$$

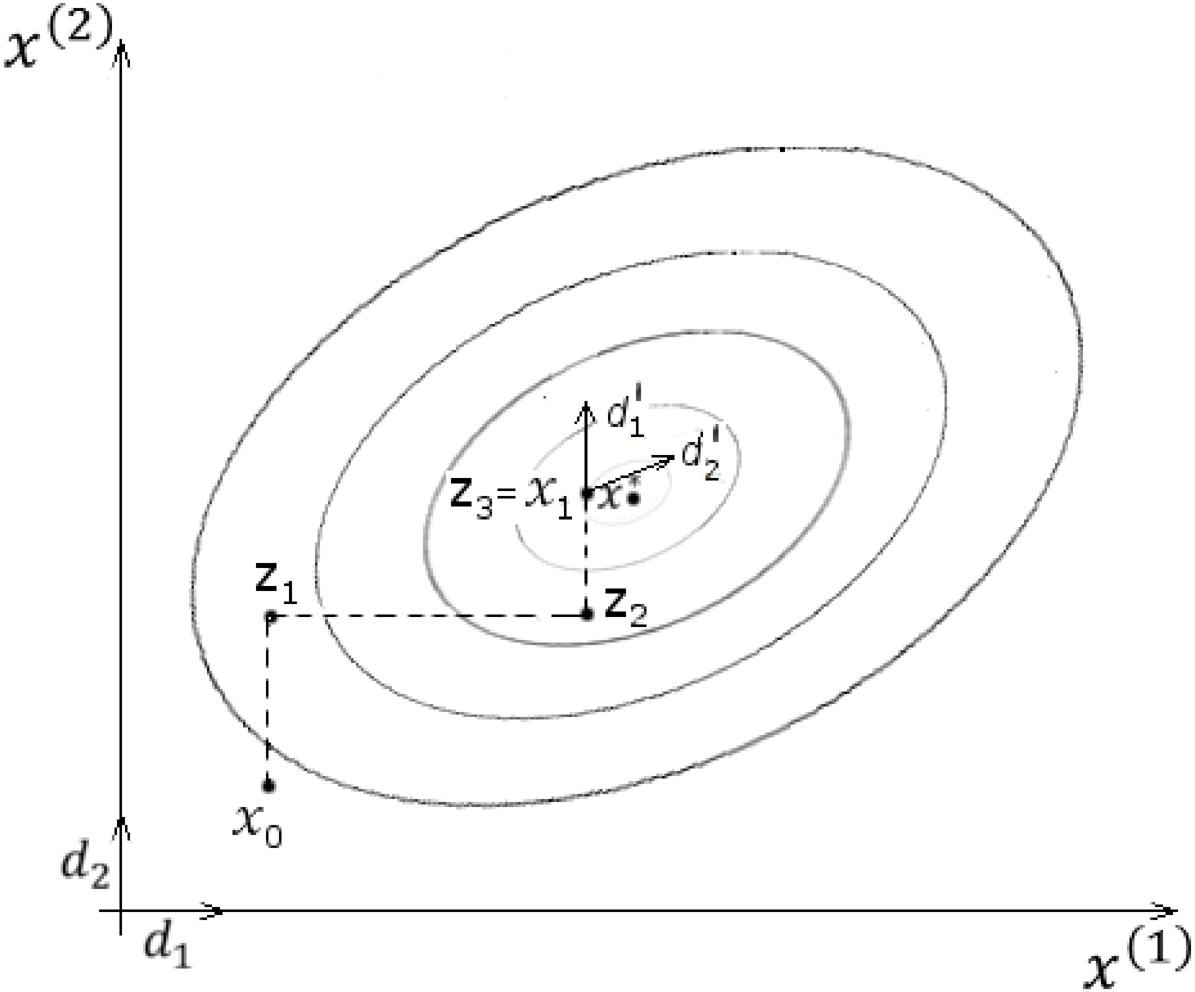


$$d' = \frac{x_1' - x_1}{\|x_1' - x_1\|}$$

$x_1 = x_0 + x^* d$
τ^* - optimal step size along the direction d from x_0
$x_1' = x_0' + \tau^{*\prime} d$
$\tau^{*\prime}$ - optimal step size along the direction d from x_0'

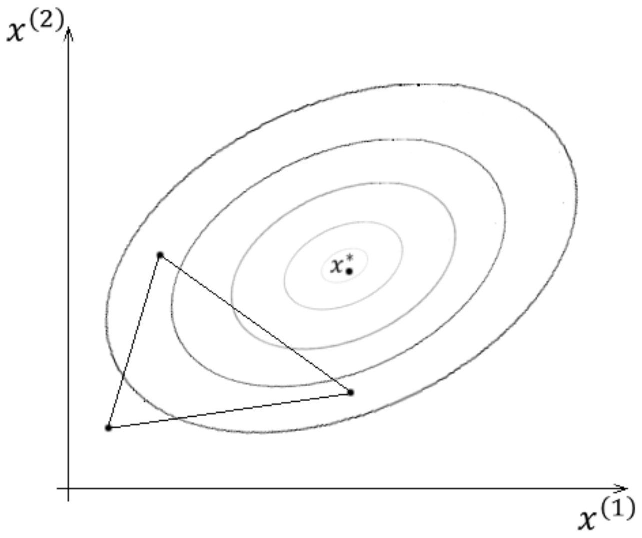
$d^T Ad' = 0$
d, d'- conjugated with respect A

Powell’s method



Nelder-Mead method

x_1 x_2 ... x_{S+1} - s-dimensional simplex



$$x_H \rightarrow F(x_H) = \max_{1 \le s \le S+1} F(x_s)$$
$$x_L \rightarrow F(x_L) = \min_{1 \le s \le S+1} F(x_s)$$
$$\bar{x} = \frac{1}{s} \sum_{s=1, s \ne H}^{S+1} x_s$$

Initial simplex:

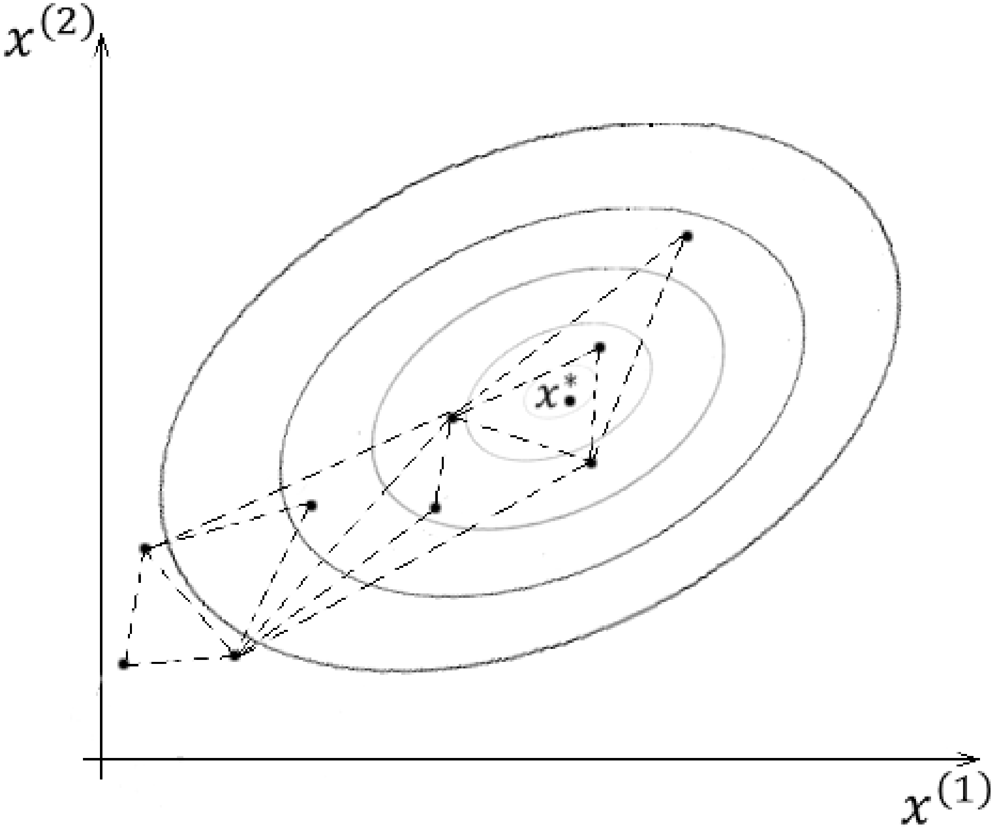
x_0, c

$$a = \frac{c}{S\sqrt{2}}(\sqrt{S+1} + \sqrt{2} - 1)$$
$$\mathrm{b} = \frac{c}{S\sqrt{2}}(\sqrt{S+1} - 1)$$

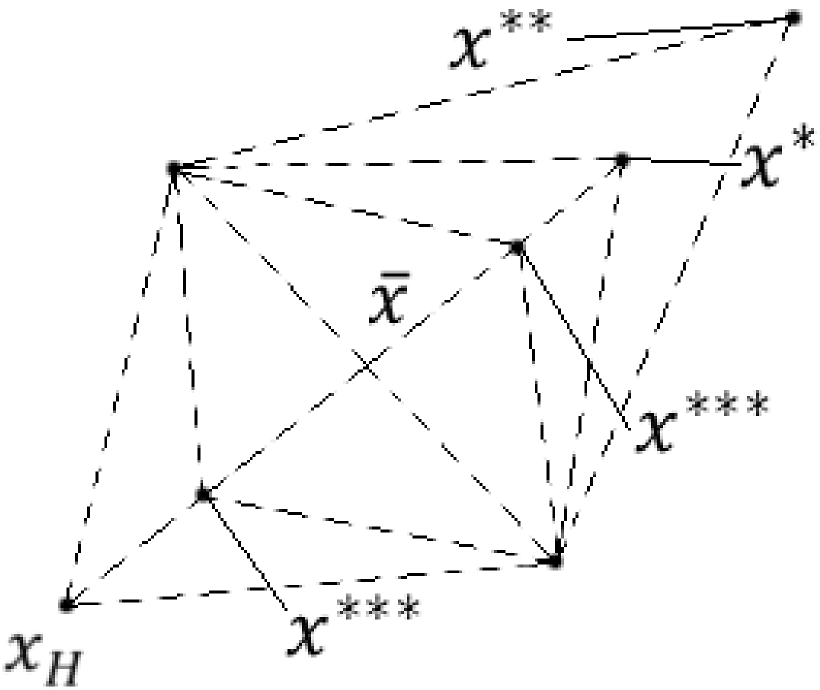
$d_j = [\quad]$

$$x_i = x_0 + d_j, x_{S+1} = x_0$$

Nelder-Mead method



Nelder-Mead method



Reflection

$x^* = \bar{x} + \alpha(\bar{x} - x_H)$

α – reflection coefficient

If $\alpha > 0$

$F(x^*) < F(x_L)$

Expansion

$x^{**} = \bar{x} + \gamma(x^* - \bar{x}) \qquad \gamma > 1$

γ – expansion coefficient

If $F(x^*) > F(x_H)$

Contraction

$x^{***} = \bar{x} + \beta(x_H - \bar{x})$

If $F(x^*) > \max\limits_{\substack{1 \le s \le S+1 \\ s \ne H}} F(x_s)$

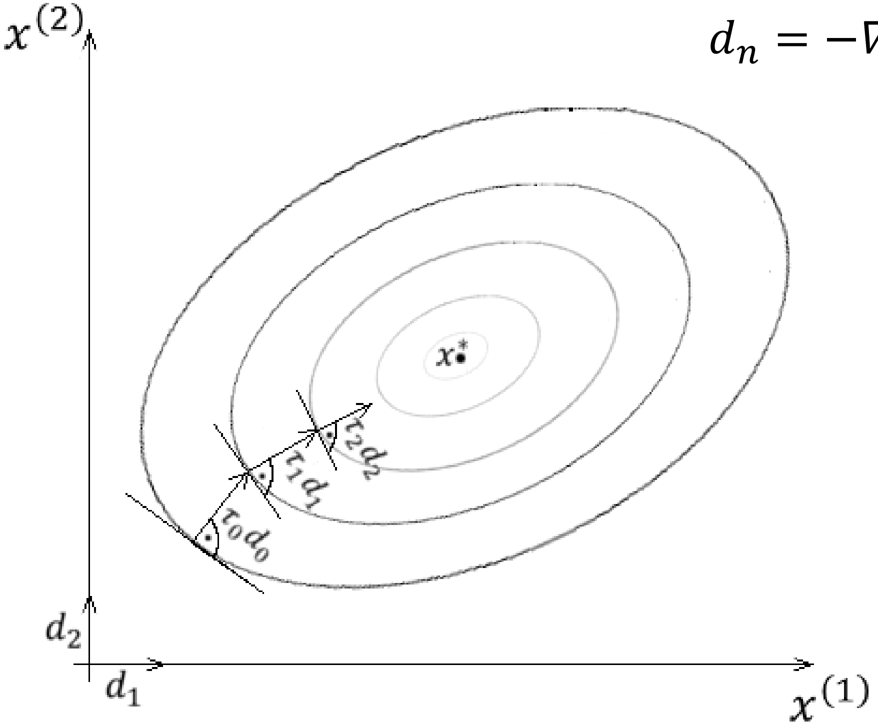
$x^{***} = \bar{x} + \beta(x^* - \bar{x}) \qquad \beta \in (0,1)$

β – contraction coefficient

The gradient descent method

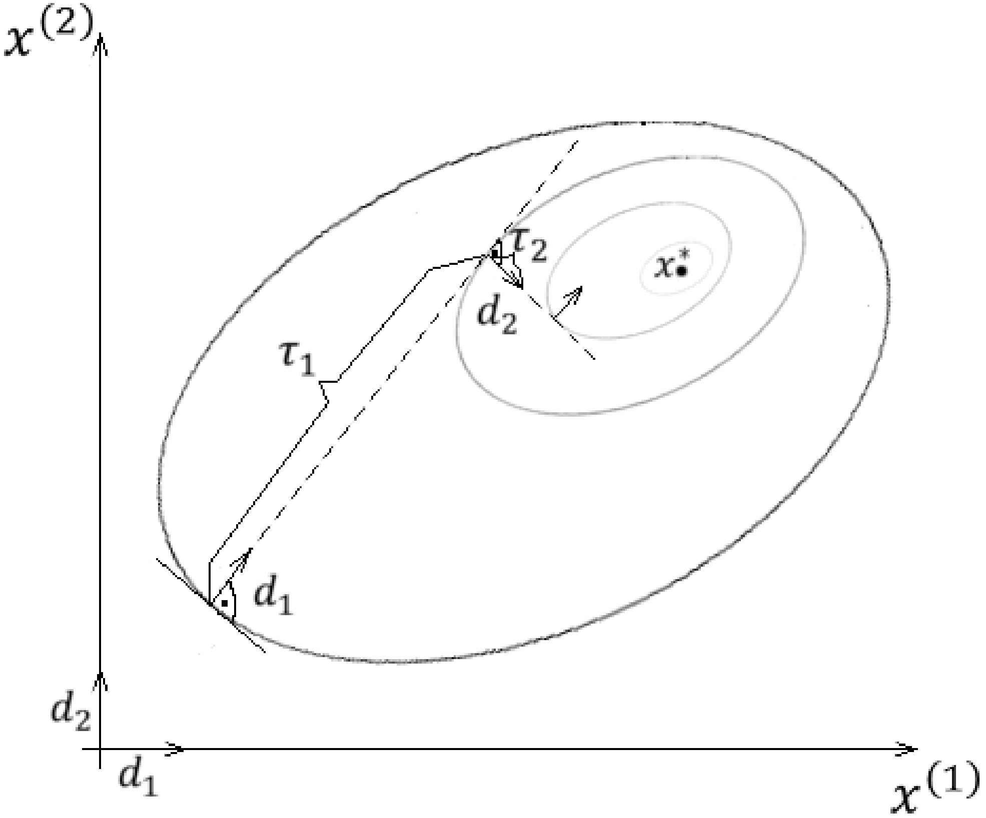
$$x_{n+1} = x_n + \tau_n d_n$$

$$d_n = -\nabla_x F(x_n)\ ; \tau_n > 0, \lim_{n\to\infty} \tau_n = \tau, \sum_{n=0}^{\infty} \tau_n = \infty$$



$$\|x_{n+1} - x_n\| = \|\tau_n d_n\| < \varepsilon$$

The gradient descent method



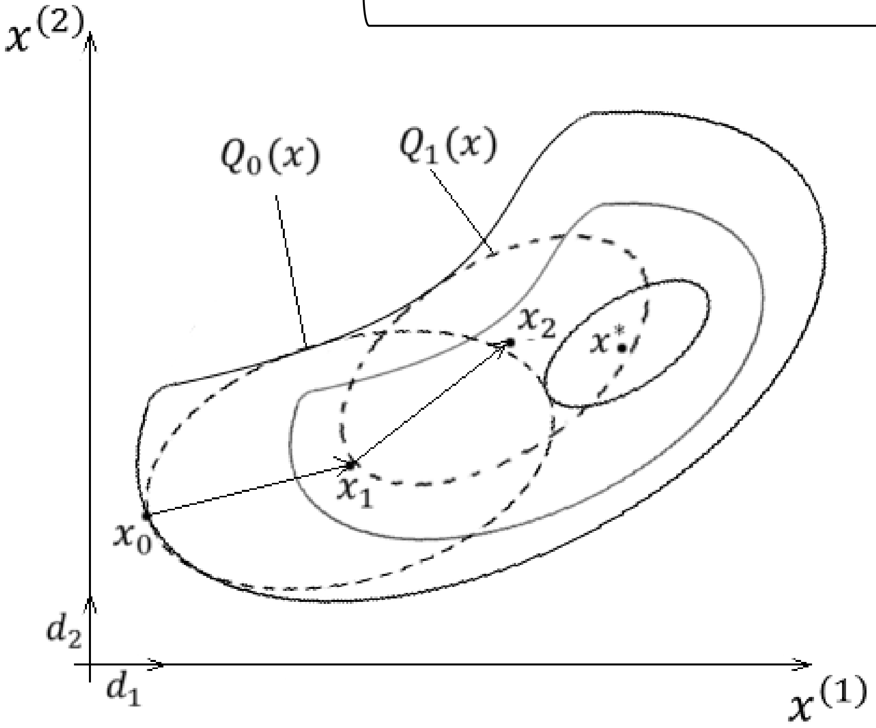
$$x_{n+1} = x_n + \tau_n d_n$$

$d_n = -\nabla_x F(x_n)$, τ_n – optimal step size along the direction d_n

$$\|x_{n+1} - x_n\| < \varepsilon$$

Newton's method

$$F(x) = \underbrace{F(x_0) + (x - x_0)^T \nabla_x F(x_0) + \frac{1}{2}(x - x_0)^T H(x_0)(x - x_0)}_{Q\quad(x)} + 0_3(\|x - x_0\|)$$



$$\nabla_x Q(x) = \nabla_x F(x_0) + H(x_0)(x^* - x_0) = O_S$$

$$x^* = x_0 - H^{-1}(x_0)\nabla_x F(x_0)$$

$$x_{n+1} = x_n - H^{-1}(x_n)\nabla_x F(x_n)$$

Variable metric methods

Step 0: $z_1 = x_0$

$d_1 = -D_1 \nabla_x F(z_1) \quad D_1 = I$

Step 1: $z_{s+1} = z_s + \tau_s d_s$ $\qquad \tau_s$ – optimal step size along the direction d_s

If $\|\tau_s d_s\| < \varepsilon$ (STOP)

otherwise go to 2

Step 2: $d_{s+1} = -D_{s+1} \nabla_x F(z_{s+1})$

$$D_{s+1} = D_s + \frac{p_s p_s^T}{p_s^T q_s} - \frac{D_s q_s q_s^T D_s}{q_s^T D_s q_s},$$

$$p_s = \tau_s d_s, \quad q_s = \nabla_x F(z_{s+1}) - \nabla_x F(z_s)$$

$s := s + 1,$ go to 1

$D_{S+1} \approx H^{-1}(x_{S+1})$

Fletcher-Reeves method of conjugate gradients

Step 0: $\mathrm{z}_1 = \mathrm{x}_0,\ s = 1,\ d_1 := -\nabla_x F(z_1)$

Step 1: $z_{s+1} := z_s + \tau_s d_s$

$\tau_s \rightarrow$ optimal step size along the direction d_s

If $\|\tau_s d_s\| < \varepsilon$ (STOP)

otherwise go to 2

Step 2: $d_{s+1} := -\nabla_x F(z_{s+1}) + \frac{\|\nabla_x F(z_{s+1})\|}{\|\nabla_x F(z_s)\|} d_s$

$s := s + 1,$ go to 1

$d_1, d_2, \ldots, d_S$ – conjugate directions

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Fletcher-Reeves method of conjugate gradients

Step 0: $\mathrm{z}_1 = \mathrm{x}_0,\ s = 1,\ d_1 := -\nabla_x F(z_1)$

Step 1: $z_{s+1} := z_s + \tau_s d_s$

$\tau_s \rightarrow$ optimal step size along the direction d_s

If $\|\tau_s d_s\| < \varepsilon$ (STOP)

otherwise go to 2

Step 2: $d_{s+1} := -\nabla_x F(z_{s+1}) + \frac{\|\nabla_x F(z_{s+1})\|}{\|\nabla_x F(z_s)\|} d_s$

$s := s + 1,$ go to 1

$d_1, d_2, \dots, d_S$ – conjugate directions

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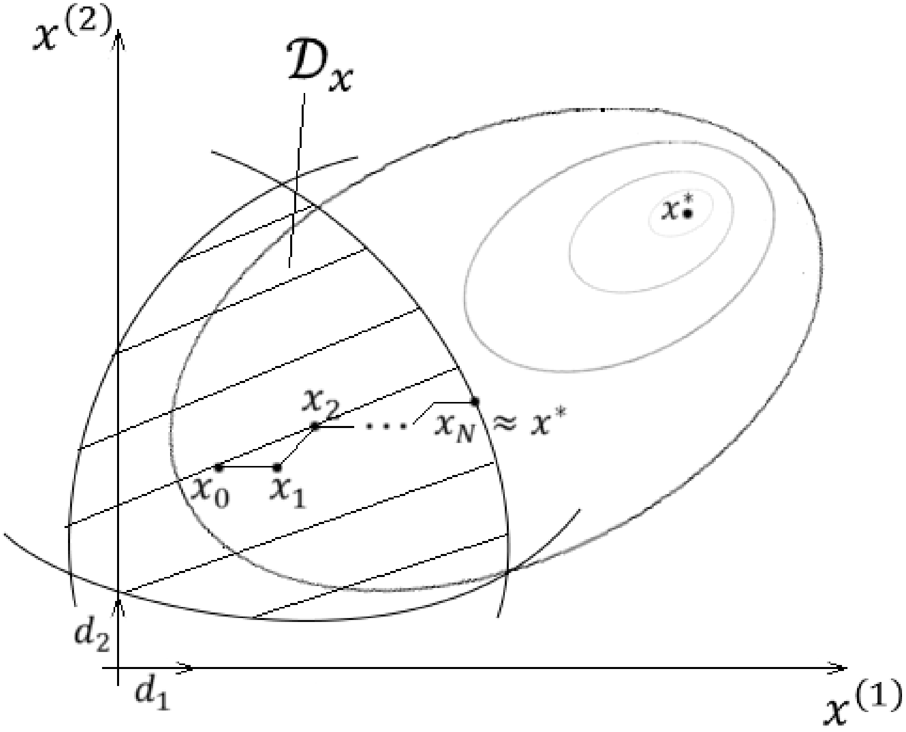
Numerical constrained optimization methods



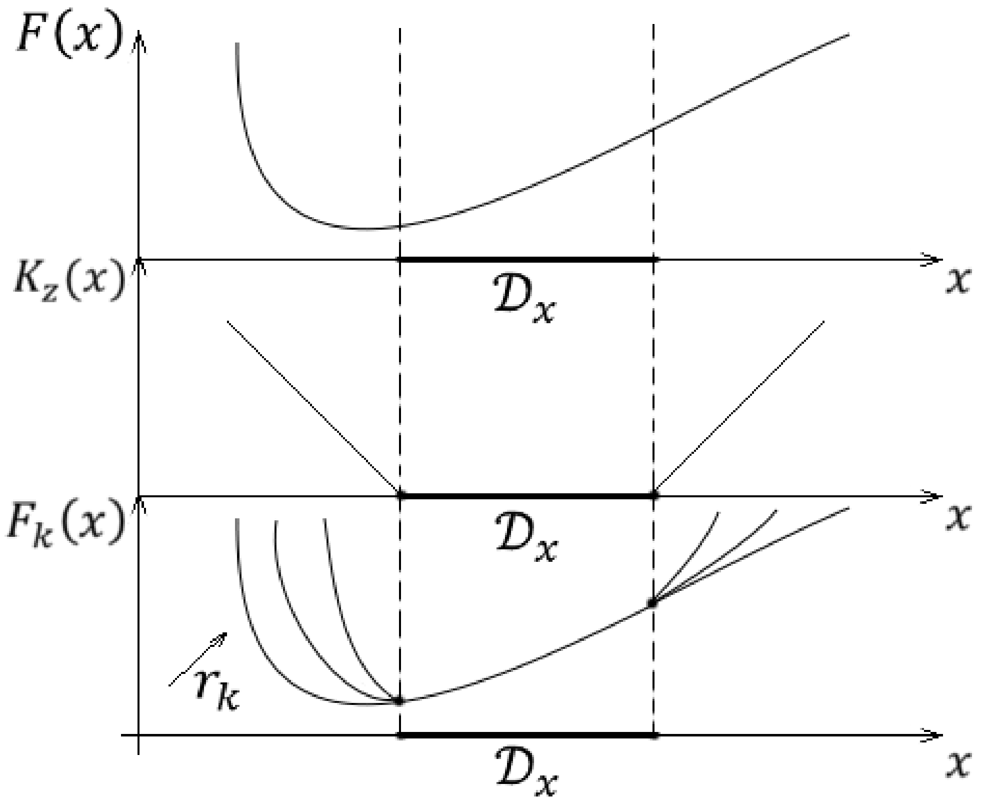
$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

1. Elimination of constraints
2. Penalty function method
    - exterior penalty
    - barrier function
3. Methods of feasible directions
4. Other approaches

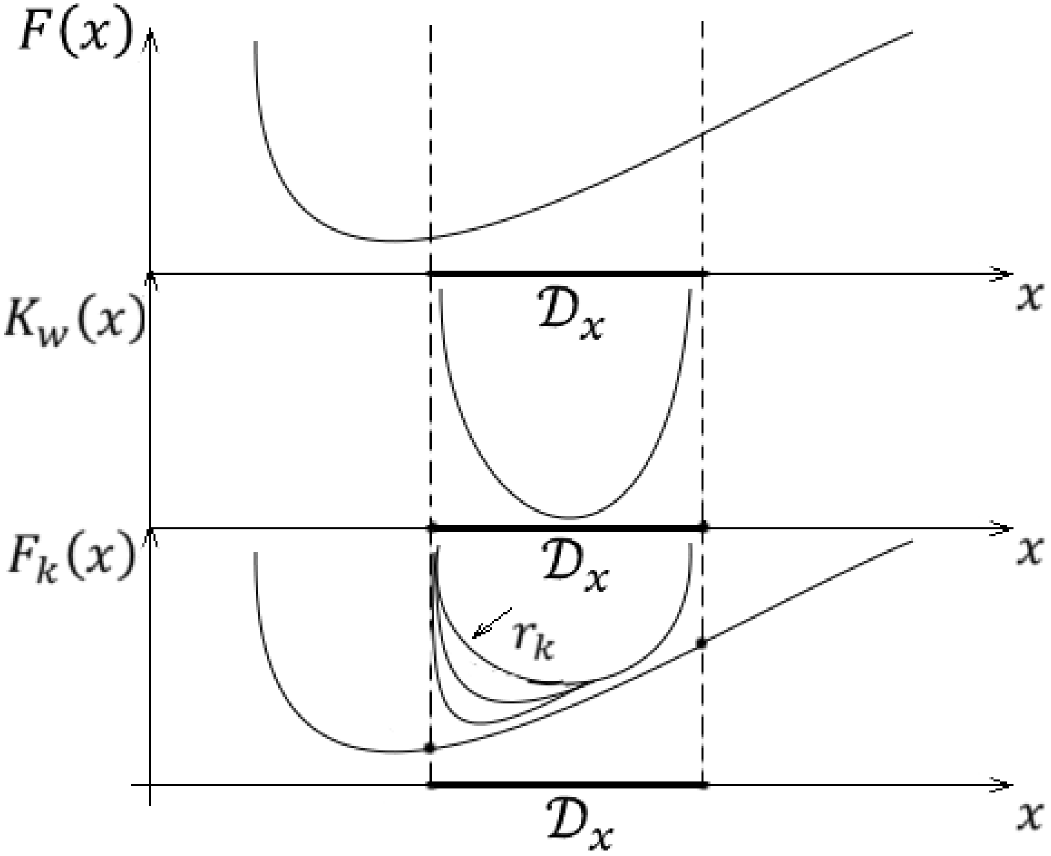
Elimination of constraints

$$\begin{tikzcd}
x^* \arrow[r] & \min\limits_{x \in D_x} F(x) \arrow[d] \\
z^* \arrow[u] & \min\limits_{z \in R^s} \bar{F}(z) \arrow[l]
\end{tikzcd}$$

$$p: R^s \to \mathcal{D}_x$$
$$x = p(z)$$
$$\bar{F}(z) = F\big(p(z)\big)$$
$$z \in R^s \to x \in \mathcal{D}_x$$



$r_k > 0$ $\qquad$ $\lim\limits_{k\to\infty} r_k = \infty$



$$r_k > 0 \qquad \lim_{k\to\infty} r_k = 0$$

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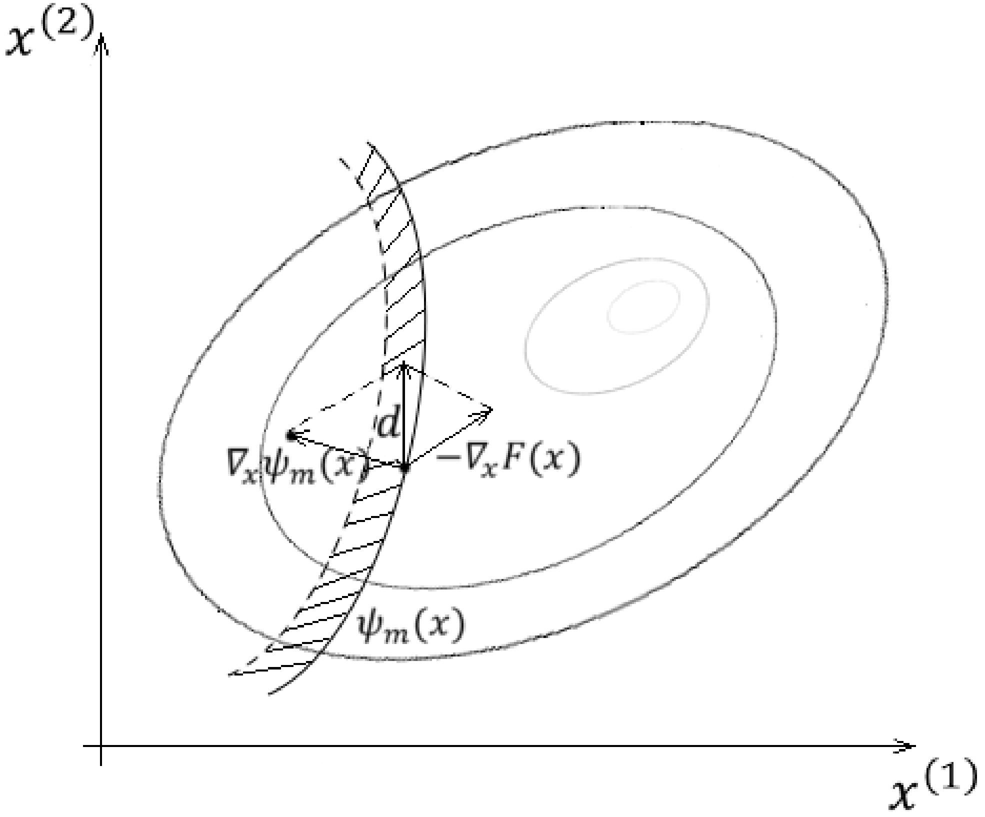
Feasible directions method



$$d = \frac{\nabla_x \psi_m(x)}{\|\nabla_x \psi_m(x)\|} - \frac{\nabla_x F(x)}{\|\nabla_x F(x)\|}$$

$$x: \psi(x) - \delta \leq 0$$

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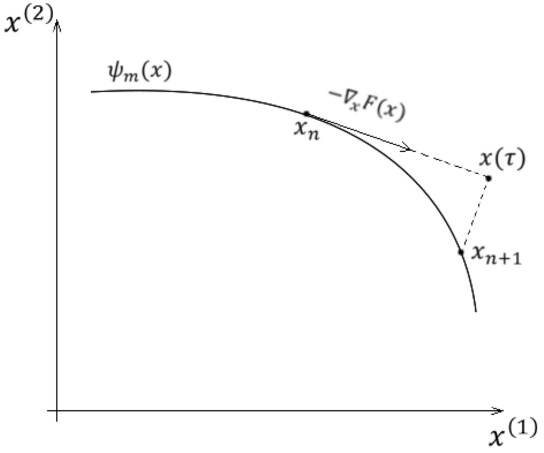
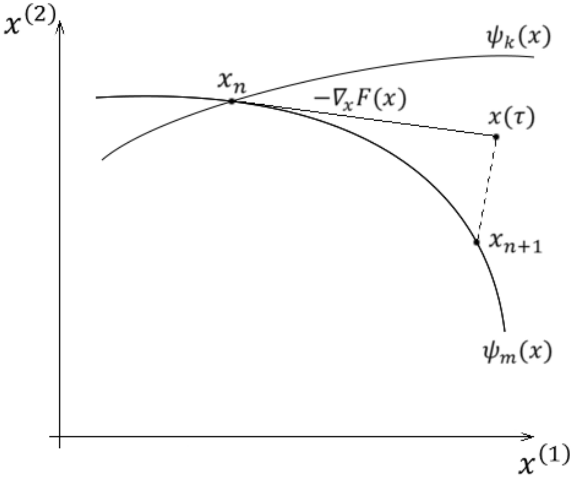
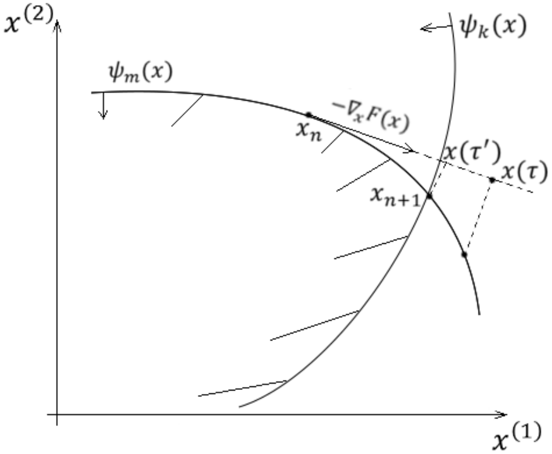
Gradient projection method of Rosen







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Random search - Down Hill method

Data: $F(x), x_0, D_x, N$

Step 0: n=0, $x^* = x_n$

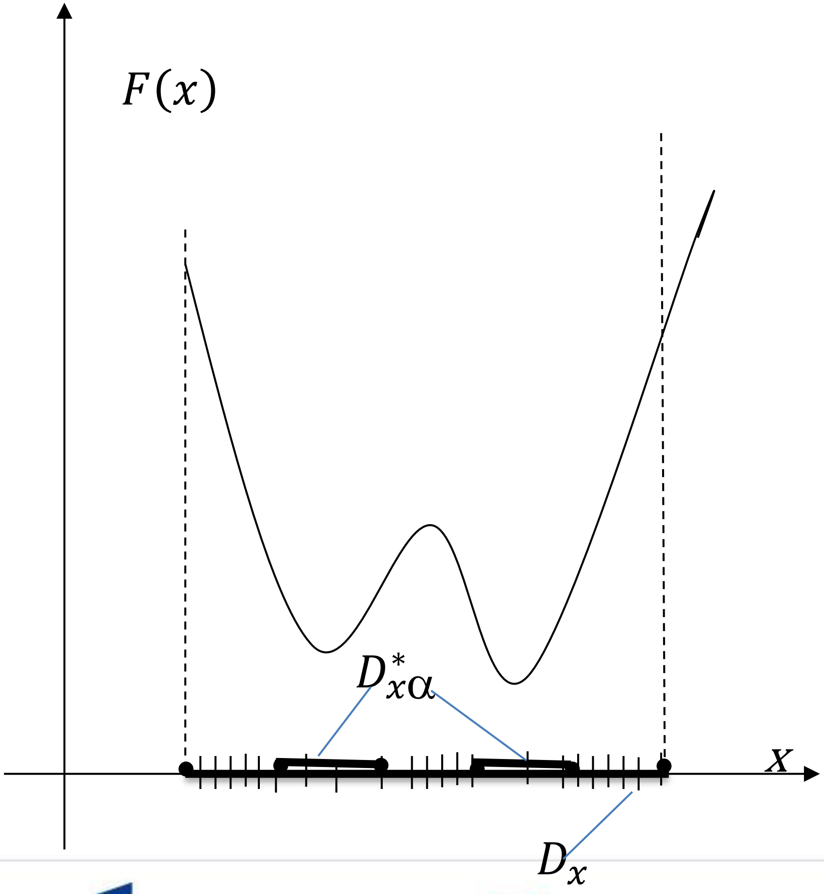
Step 1: Generate point x_{n+1} in the set D_x with unity probability density

Step 2: IF $F(x_{n+1}) < F(x^*)$ THEN $x^* = x_{n+1}$

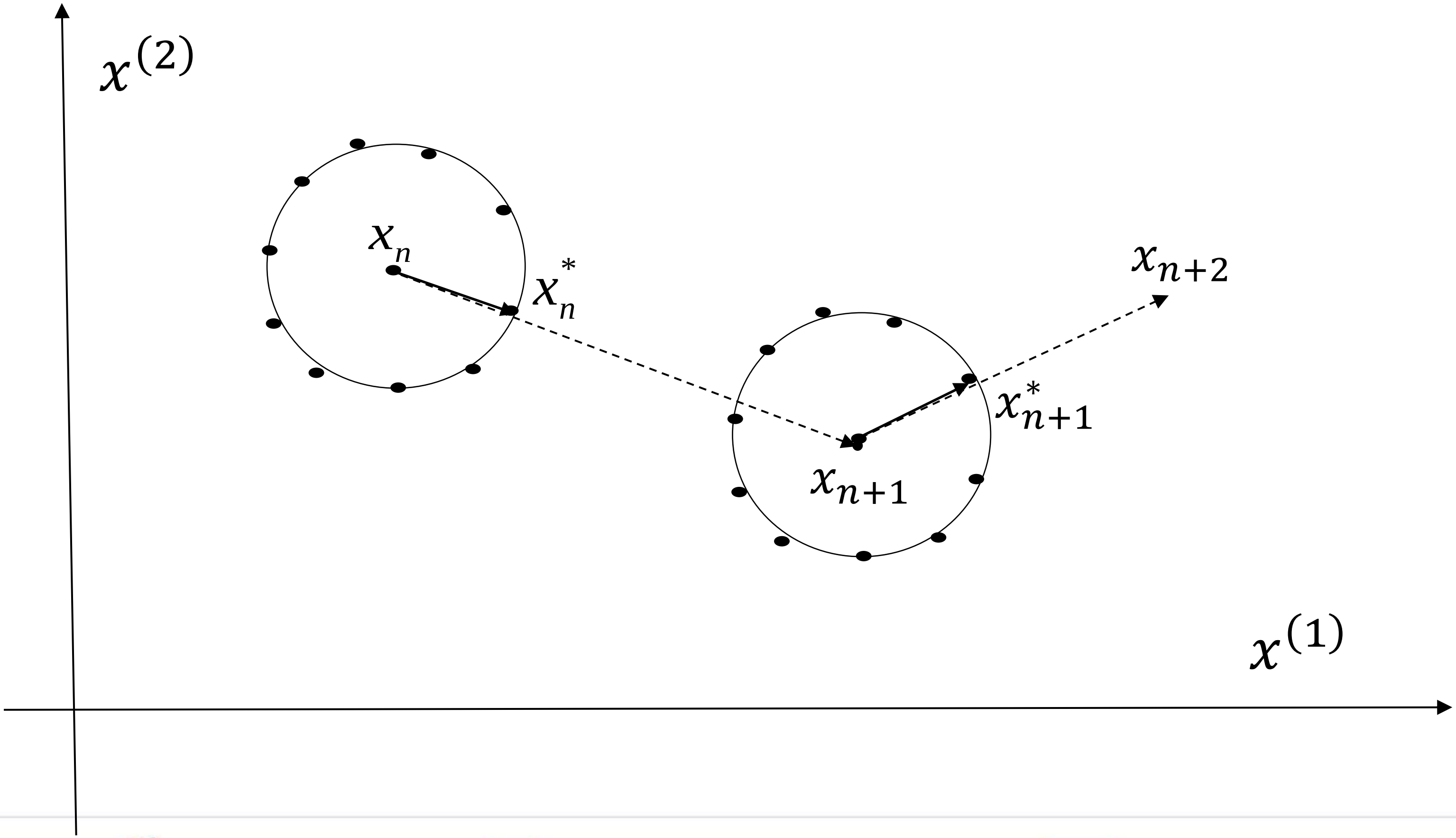
Step 3: IF $n < N$ THEN $n = n + 1$ GO TO STEP 1

Step 4: $x^* = x_N$

Random search



$$f(x)=\frac{F(x)}{\int\limits_{D_x} F(x)dx}$$


$x^{(2)}$
x_n
x_n^*
x_{n+2}
x_{n+1}^*
x_{n+1}
$x^{(1)}$

Nature-Inspired Algorithms Bibliogrphy

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- Automatic Tuning of a Retina Model for a Cortical Visual Neuroprosthesis Using a Multi-Objective Optimization Genetic Algorithm, Antonio Martínez-Álvarez, Rubén Crespo-Cano, Ariadna Díaz-Tahoces et. al., International Journal of Neural Systems 26/7, 2016

General problem formulation

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$\mathcal{D}_x = \{x \in R^s, \varphi_l(x) = 0, l = 1, 2, \dots, L, \psi_m(x) \leq 0, m = 1, 2, \dots, M\}$$

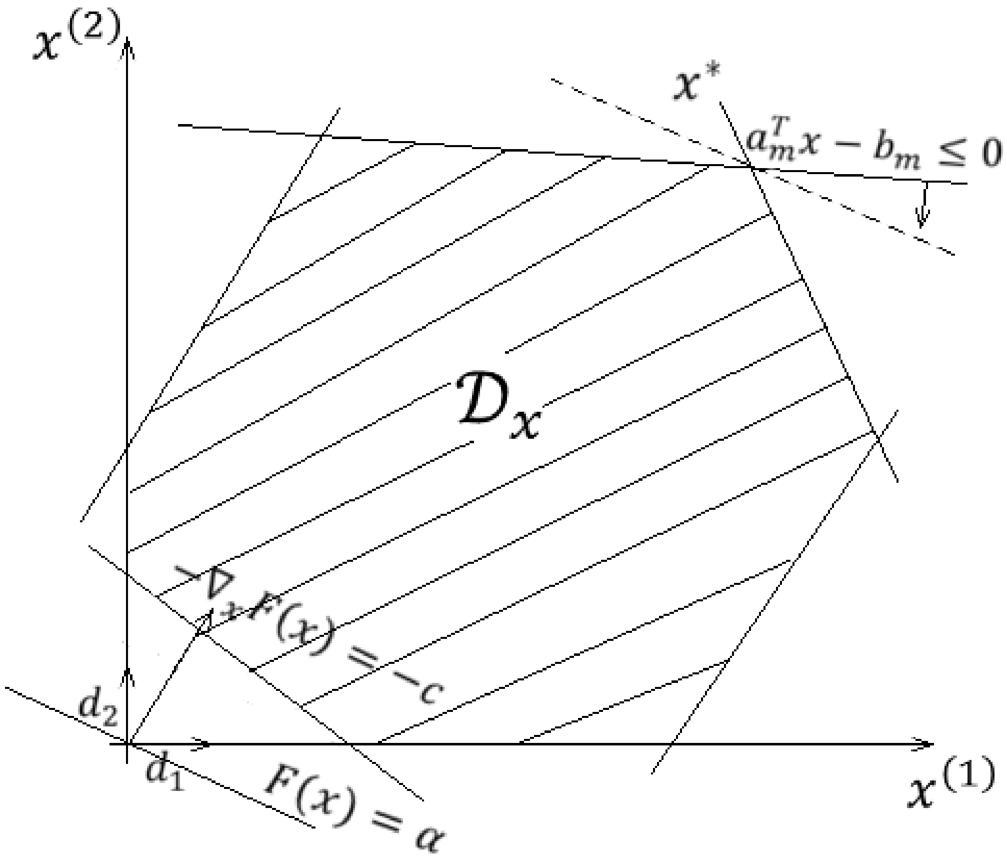
$$F(x) = c^T x = \sum_{s=1}^{S} c_s x^{(s)}$$

$$\varphi_l(x) = a_l^T - b_l = \sum_{s=1}^{S} a_{ls} x^{(s)} - b_l = 0 \quad l = 1, 2, \dots, L$$

$$\psi_m(x) = a_m^T x - b_m \leq 0 = \sum_{s=1}^{S} a_{ms} x^{(s)} - b_m \leq 0 \quad m = 1, 2, \dots, M$$

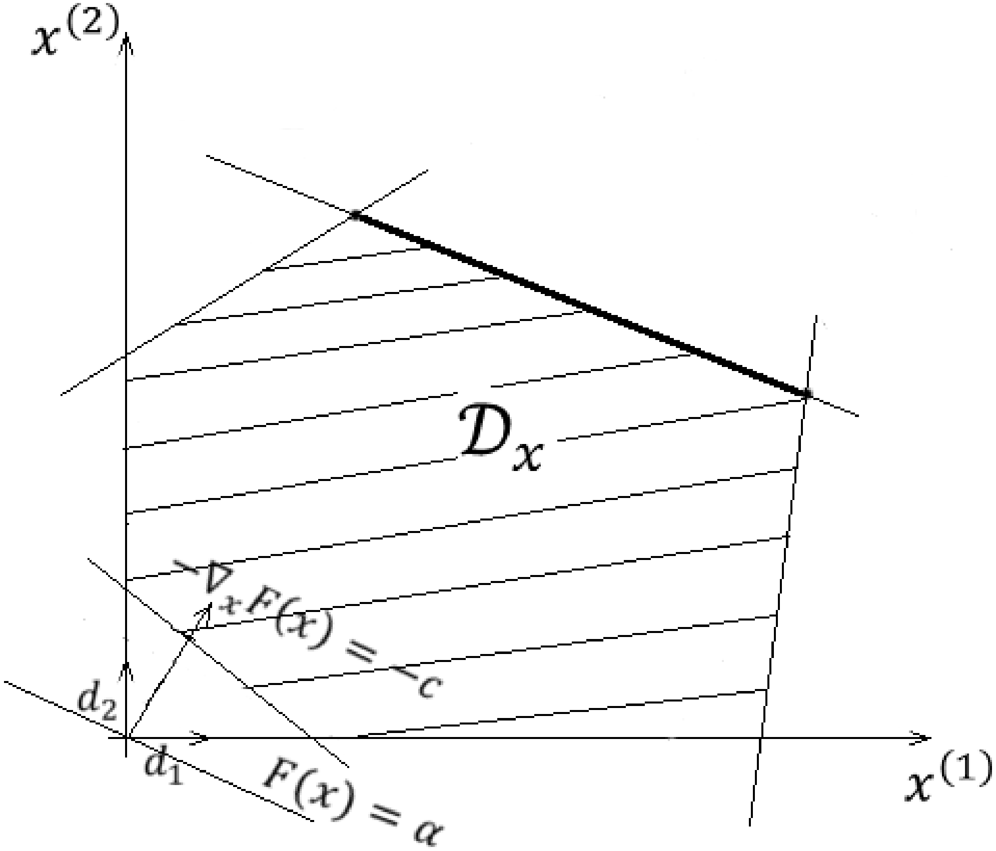
$$x^{(s)} \geq 0 \quad s = 1, 2, \dots, S$$

Geometric view



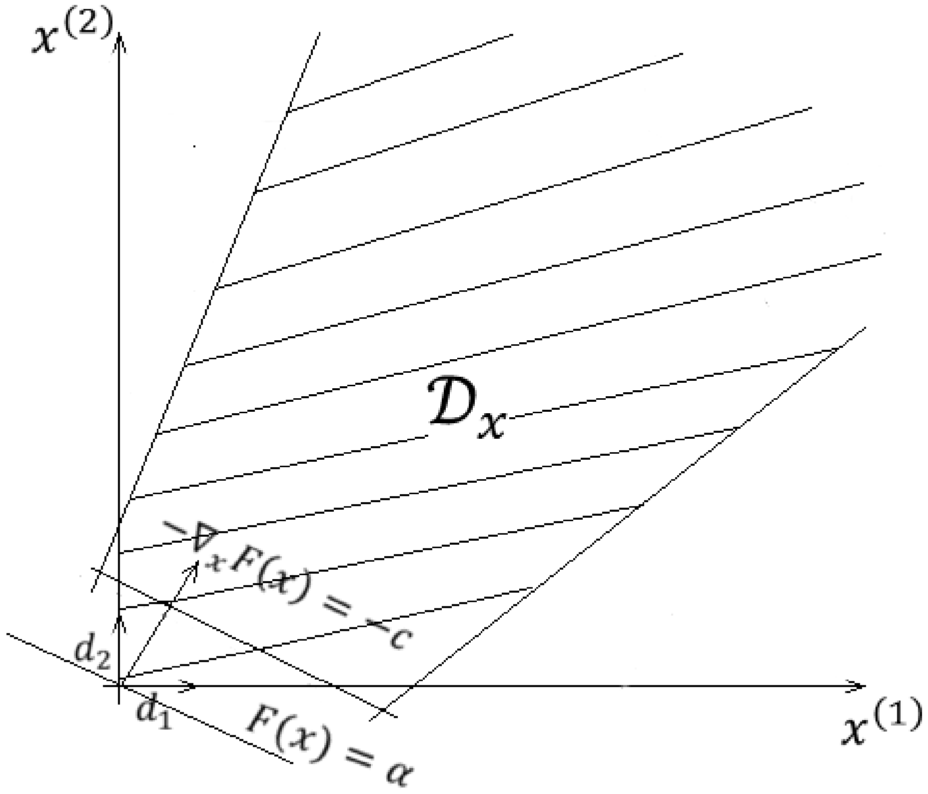
1. Solution is located on a vertex

Geometric view



2. Solution is located on an edge

Geometric view



3. Unbounded solution

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Standard form

$$F(x) \; = c^T x$$

A: $\mathcal{D}_X = \{x \in R^S, Ax - b = 0_L, x \geq 0_S\}$

or

B: $\mathcal{D}_x = \{x \in R^S, Ax - b \leq 0_L, x \geq 0_S\}$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_S \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_L \end{bmatrix}, \quad x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(S)} \end{bmatrix}, \quad A_{SxL} = \begin{bmatrix} a_{11} & \cdots & a_{1S} \\ \vdots & \ddots & \vdots \\ a_{L1} & \cdots & a_{LS} \end{bmatrix}$$

The simplex method

1. Generation of initial basis
2. Checking $c - c_B B^{-1} A \geq 0_S$. If it holds, then x_B is basic feasible solution $x = [x_B \; 0]$
3. Such a k that $c_k - z_k = \min_{1 \leq s \leq S} (c_s - z_s)$ is introduced to the basis
4. Checking, whether $h_k \leq 0$, if it holds true – solution is unbounded
5. Removing such *l* from the basis, for which:

$$\frac{h_{l0}}{h_{lk}} = \min_{1 \leq s \leq S} \{\frac{h_{s0}}{h_{sk}}, h_{sk} > 0\}$$

6. $I_B := I_B \setminus \{l\} \cup \{k\}$

   $I_B = \{j \in \{1, 2, \dots, S\}$ $\quad x^{(j)}$ belongs to the basis }

				$\boldsymbol{c_1}$	$\cdots$	$\boldsymbol{c_k}$	$\cdots$	$\boldsymbol{c_S}$			
	Zmienne bazowe	c_B	h_0	h_1	…	h_k	…	h_S		$\frac{h_{s0}}{h_{sk}}$	$h_{sk} \geq 0$
	x_{j1}	c_{j1}	h_{10}	h_{11}	$\cdots$	h_{1k}	$\cdots$	h_{1S}			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$\leftarrow$	x_{jl}	c_{jl}	h_{l0}	h_{l1}	$\cdots$	$\enclose{circle}{h_{lk}}$	$\cdots$	h_{lS}			
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			
	x_{jL}	c_{jL}	h_{L0}	h_{L1}	$\cdots$	h_{Lk}	$\cdots$	h_{LS}			
				$c_1 - z_1$	$\cdots$	$c_k - z_k$	$\cdots$	$c_S - z_S$			
						$\uparrow$					

$$z_k = \sum_{s \in I_B} c_s h_{sk}$$

$$h'_{ls} := \frac{h_{ls}}{h_{lk}}; \quad h'_{is} = h_{is} - \frac{h_{ik} h_{ls}}{h_{lk}}$$

$$s = 1, 2, \dots, S \qquad i = 0, 1, \dots, S$$

$$s \in I_B \backslash \{l\}$$

Quadratic programming

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

$$F(x) = x^T D x + c^T x$$

$$D_x = \{x \in R^s, Ax = b, x \geq 0\}$$

Linear Fractional Programming

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F(x) = \frac{a^T x + b}{c^T x + d} \qquad a \in \mathcal{R}^S, b \in \mathcal{R}, c \in \mathcal{R}^S, d \in \mathcal{R}$$

$$c^T x + d \neq 0$$

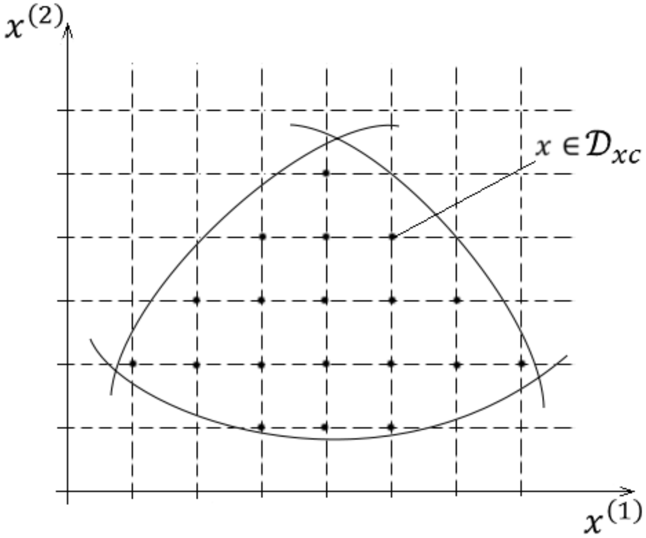
$$\mathcal{D}_x = \{x \in R^S, Ax - e \leq 0_L, x \geq 0_S\}$$

Charnes - Cooper Method

Discrete programming – branch and bound method

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_{xc}} F(x)$$

$\mathcal{D}_{xc} = \mathcal{D}_x \cap \{x^{(s)} \subset C \quad s = 1, 2, \ldots, S\}$ integer decision variables



Special case

$\mathcal{D}_{xc} = \{x_1, x_2, \ldots, x_k\}$ – finite set, k – large number

$\mathcal{D}_{xc} = \{0,\ 1\}$ - binary programming

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Step 0: $\mathcal{D}_0 = \{\mathcal{D}_{xc} = \mathcal{D}_{01}\}, n = 0, J_0 = 1$

Step 1: Determine a set $\mathcal{D}^* \in \mathcal{D}_n$

$$\mathcal{F}(\mathcal{D}^*) = \min_{\mathcal{D}\in\mathcal{D}_n} \mathcal{F}(\mathcal{D})$$

Step 2: Checking whether $\mathcal{D}^*$ is a set ? ($\{x^*\} = \mathcal{D}^*$)
or $x^* \sim \mathcal{F}(\mathcal{D}^*)$ i.e. $\mathcal{F}(\mathcal{D}^*) = F(x^*) \quad x^* \in \mathcal{D}^*$ (?) then x^* optimal solution
STOP

Step 3: $\mathcal{D}^* = \mathcal{D}_{nk}$ is split up into M disjoint sets

$$\mathcal{D}_{1nk}\mathcal{D}_{2nk} \dots \mathcal{D}_{Mnk} \qquad \mathcal{D}_{nk} = \bigcup_{m=1}^{M} \mathcal{D}_{mnk}$$

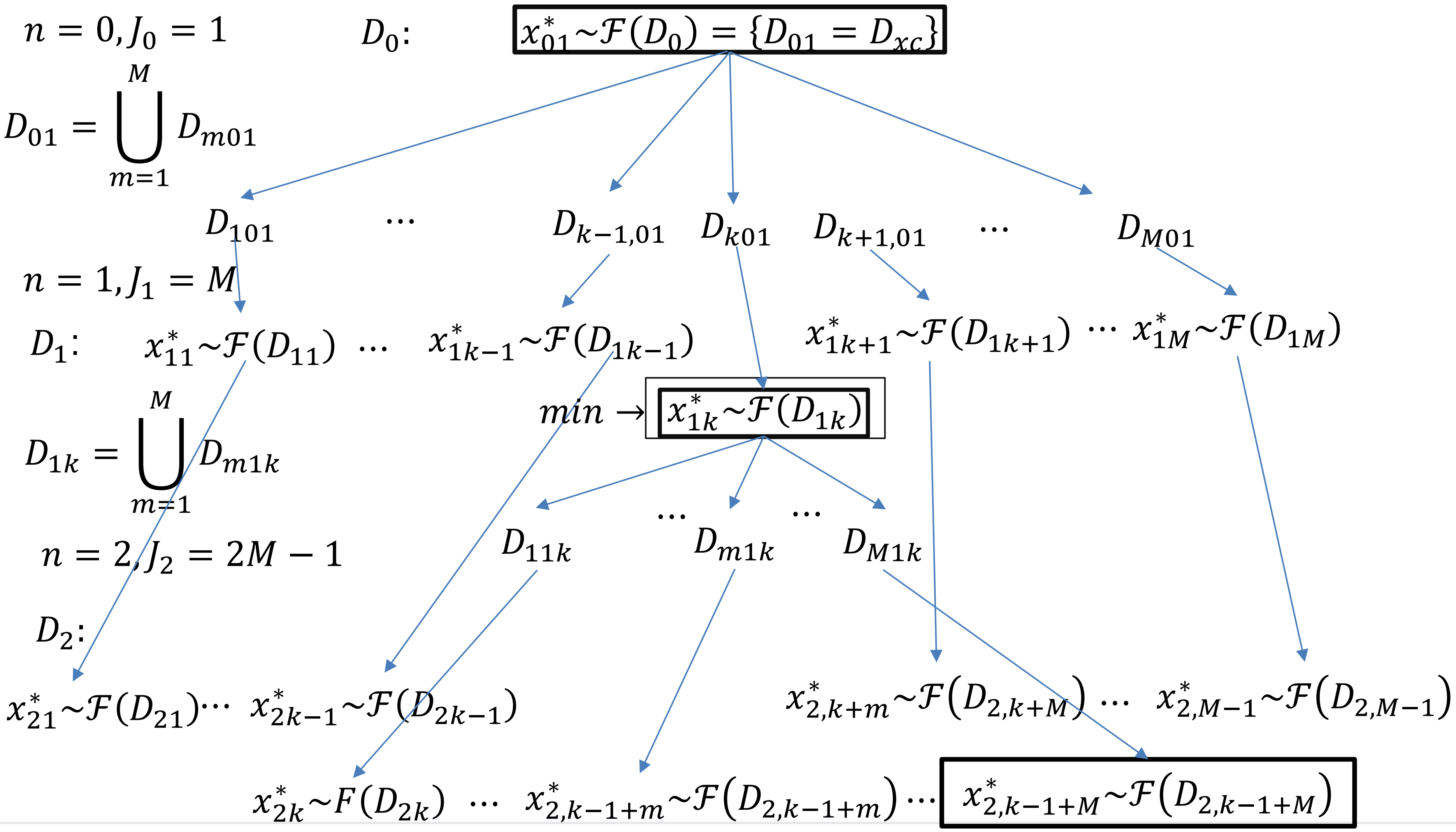
Step 4: $\mathcal{D}^* = \mathcal{D}_{nk}$

$\mathcal{D}_{n+1} = \mathcal{D}_n \cup \{\mathcal{D}_{1nk}, \mathcal{D}_{2nk}, \dots, \mathcal{D}_{Mnk}\} \backslash \mathcal{D}_{nk}$

$\mathcal{D}_{n+1,j} = \mathcal{D}_{nj} \qquad j = 1, 2, \dots, k-1$

$\mathcal{D}_{n+1,j} = \mathcal{D}_{mnk} \qquad j = k+m, m = 1, 2, \dots, M$

$\mathcal{D}_{n+1,j} = \mathcal{D}_{ni} \qquad j = k+M+i, i = k+1, \dots, J_n, J_{n+1} = J_n + M - 1$



$n = 0, J_0 = 1$

D_0: $\boxed{x_{01}^{*} \sim \mathcal{F}(D_0) = \{D_{01} = D_{xc}\}}$

$$D_{01} = \bigcup_{m=1}^{M} D_{m01}$$

D_{101} $\cdots$ $D_{k-1,01}$ D_{k01} $D_{k+1,01}$ $\cdots$ D_{M01}

$n = 1, J_1 = M$

D_1: $x_{11}^{*} \sim \mathcal{F}(D_{11})$ $\cdots$ $x_{1k-1}^{*} \sim \mathcal{F}(D_{1k-1})$ $x_{1k+1}^{*} \sim \mathcal{F}(D_{1k+1})$ $\cdots$ $x_{1M}^{*} \sim \mathcal{F}(D_{1M})$

$min \rightarrow \boxed{\boxed{x_{1k}^{*} \sim \mathcal{F}(D_{1k})}}$

$$D_{1k} = \bigcup_{m=1}^{M} D_{m1k}$$

D_{11k} $\cdots$ D_{m1k} $\cdots$ D_{M1k}

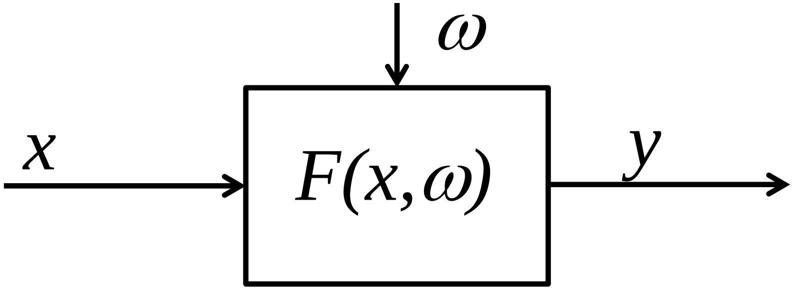
$n = 2, J_2 = 2M - 1$

D_2:

$x_{21}^{*} \sim \mathcal{F}(D_{21}) \cdots$ $x_{2k-1}^{*} \sim \mathcal{F}(D_{2k-1})$ $x_{2,k+m}^{*} \sim \mathcal{F}\left(D_{2,k+M}\right)$ $\cdots$ $x_{2,M-1}^{*} \sim \mathcal{F}\left(D_{2,M-1}\right)$

$x_{2k}^{*} \sim F(D_{2k})$ $\cdots$ $x_{2,k-1+m}^{*} \sim \mathcal{F}\left(D_{2,k-1+m}\right) \cdots$ $\boxed{x_{2,k-1+M}^{*} \sim \mathcal{F}\left(D_{2,k-1+M}\right)}$

Decision making under uncertainty



$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right]$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}\left[\mathscr{D}_x(\underline{\omega})\right] =$$

$$= \left\{ x \in \mathscr{R}^S;\ \underset{\underline{\omega}}{E}\left[\varphi_l(x,\underline{\omega})\right] = 0,\ l = 1, \ldots, L,\ \underset{\underline{\omega}}{E}\left[\psi_m(x,\underline{\omega})\right] \le 0,\ m = 1, \ldots, M \right\}$$

$$x^* \to F(x^*) = \min_{x \in D_x} F(x)$$

A game against nature

a_i - the minimum profit for *i*–th row

A_i - the maximum profit for *i*–th row

$$H_i(\gamma)=a_i\,\gamma+A_i(1-\gamma)\quad \gamma\in[0,1]$$

The Hurwitz rule.

Analyzing the subsequent rows of the matrix we find the minimum and the maximum revenue, i.e. values a_i, A_i and value of the function $H_i(\gamma)$ for a given γ. We make such a decision, for which the value of the function $H_i(\gamma)$ is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

Type of corn	Weather conditions			*a* min	*A* max	$H(\gamma)$ $\gamma = 0.5$
	drought	normal	rain			
1	8	10	12	8	12	10
2	10	11	7	7	11	9
3	9	13	8	8	13	10.5
4	11	10	6	6	11	8.5
5	10	10	γ 9	9	10	9.5

← max

Two-person zero-sum game

Two-player zero-sum game

Payoff matrix of player A:

B A	B_1	B_2	…	B_m	…	B_M
A_1	a_{11}	a_{12}	…	a_{1m}	…	a_{1M}
A_2	a_{21}	a_{22}	…	a_{2m}	…	a_{2M}
…	…	…	…	…	…	…
A_n	a_{n1}	a_{n2}	…	a_{nm}	…	a_{nM}
…	…	…	…	…	…	…
A_N	a_{N1}	a_{N2}	…	a_{Nm}	…	a_{NM}

Payoff matrix of player B:

B A	B_1	B_2	…	B_m	…	B_M
A_1	$-a_{11}$	$-a_{12}$	…	$-a_{1m}$	…	$-a_{1M}$
A_2	$-a_{21}$	$-a_{22}$	…	$-a_{2m}$	…	$-a_{2M}$
…	…	…	…	…	…	…
A_n	$-a_{n1}$	$-a_{n2}$	…	$-a_{nm}$	…	$-a_{nM}$
…	…	…	…	…	…	…
A_N	$-a_{N1}$	$-a_{N2}$	…	$-a_{Nm}$	…	$-a_{NM}$

Player A aims to maximize revenue

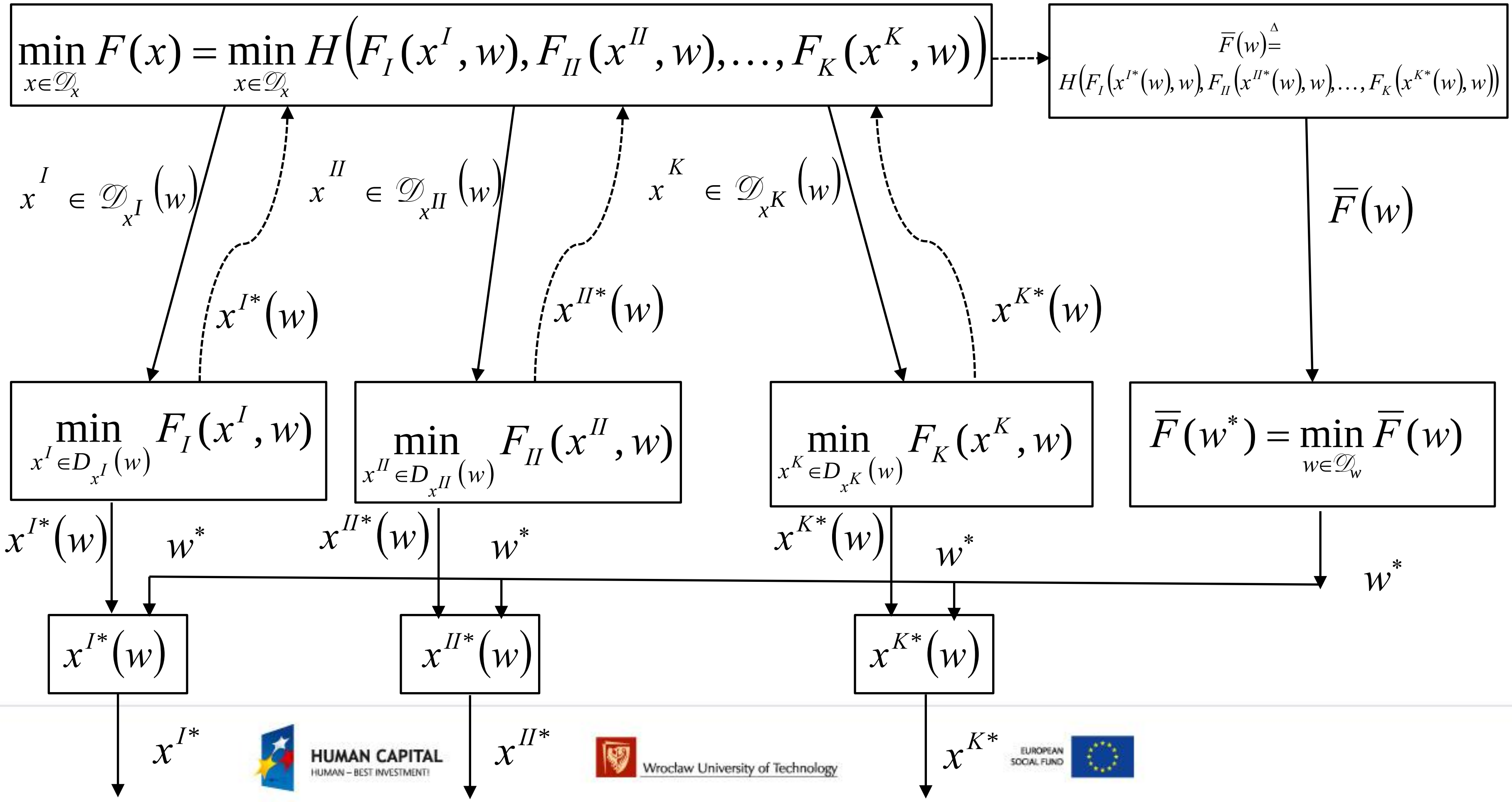
Player B aims to minimize losses

Usually the payoff matrix of player A is presented

Decision making using game theory

- Typical approaches to game solving
  - determination of saddle point
  - removal of dominated strategies
  - determination of mixed strategies for:
    - N=2 and M=2
    - N>2 and M>2

Separable goal function and separable constrains with coordinate variable



$$\min_{x\in\mathscr{D}_x} F(x) = \min_{x\in\mathscr{D}_x} H\left(F_I(x^I,w), F_{II}(x^{II},w),\ldots,F_K(x^K,w)\right)$$

$$\bar{F}(w) \overset{\Delta}{=} H\left(F_I\left(x^{I*}(w),w\right), F_{II}\left(x^{II*}(w),w\right),\ldots,F_K\left(x^{K*}(w),w\right)\right)$$

$x^I \in \mathscr{D}_{x^I}(w)$

$x^{I*}(w)$

$x^{II} \in \mathscr{D}_{x^{II}}(w)$

$x^{II*}(w)$

$x^K \in \mathscr{D}_{x^K}(w)$

$x^{K*}(w)$

$\bar{F}(w)$

$$\min_{x^I\in D_{x^I}(w)} F_I(x^I,w)$$

$$\min_{x^{II}\in D_{x^{II}}(w)} F_{II}(x^{II},w)$$

$$\min_{x^K\in D_{x^K}(w)} F_K(x^K,w)$$

$$\bar{F}(w^*) = \min_{w\in\mathscr{D}_w} \bar{F}(w)$$

$x^{I*}(w)$ w^*

$x^{II*}(w)$ w^*

$x^{K*}(w)$ w^*

w^*

$x^{I*}(w)$

$x^{II*}(w)$

$x^{K*}(w)$

x^{I*}

x^{II*}

x^{K*}

Multistage optimization

Step 1. $x_S^* = G_S\left(x_1,\ldots,x_{S-1}\right) \to F_S\left(x_1, x_2,\ldots,x_S^*\right) = \min_{x_S \in \mathscr{D}_{xS}} F_S\left(x_1,x_2,\ldots,x_S\right)$

The value of the goal function in the optimal solution:

$$F_{S-1}\left(x_1, x_2,\ldots,x_{S-1}\right) \overset{\Delta}{=} F_S\left(x_1, x_2,\ldots,x_S^*\right) = F_S\left(x_1, x_2,\ldots,G_S\left(x_1,\ldots,x_{S-1}\right)\right)$$

Constraints in the optimal solution:

$$\mathscr{D}_{xS-1}\left(x_1,\ldots,x_{S-1}\right) \overset{\Delta}{=} \mathscr{D}_{xS}\left(x_1,\ldots,x_{S-1}, x_S^* = G_S\left(x_1,\ldots,x_{S-1}\right)\right) =$$

$$\left\{\begin{array}{l} \left[x_1 \ x_2 \ \cdots \ x_{S-1}\right]^T \in \mathscr{R}^{S-1} : \\ \varphi_{lS}\left(x_1, x_2, \cdots, G_S\left(x_1,\ldots,x_{S-1}\right)\right) = \varphi_{lS-1}\left(x_1, x_2, \cdots, x_{S-1}\right) = 0, \, l = 1, 2, \ldots, L, \\ \psi_{mS}\left(x_1, x_2, \cdots, G_S\left(x_1,\ldots,x_{S-1}\right)\right) = \psi_{mS-1}\left(x_1, x_2, \cdots, x_{S-1}\right) \le 0, \, m = 1, 2, \ldots, M \end{array}\right\}$$

Multistage optimization

Step 2. $x^*_{S-1} = G_{S-1}(x_1,\ldots,x_{S-2}) \to F_{S-1}(x_1, x_2,\ldots,x^*_{S-1}) = \min_{x_{S-1}\in\mathcal{D}_{xS-1}} F_{S-1}(x_1,x_2,\ldots,x_{S-1})$

The value of the goal function in the optimal solution:

$$F_{S-2}(x_1, x_2,\ldots,x_{S-2}) \overset{\Delta}{=} F_{S-1}(x_1, x_2,\ldots,x^*_{S-1}) = F_{S-1}(x_1, x_2,\ldots,G_{S-1}(x_1,\ldots,x_{S-2}))$$

Constraints in the optimal solution:

$$\mathcal{D}_{xS-2}(x_1,\ldots,x_{S-2}) \overset{\Delta}{=} \mathcal{D}_{xS-1}(x_1,\ldots,x_{S-2},x^*_{S-1} = G_{S-1}(x_1,\ldots,x_{S-2})) =$$

$$\left\{\begin{array}{l} [x_1\ x_2\ \cdots\ x_{S-2}]^T \in \mathcal{R}^{S-2}: \\ \varphi_{lS-1}(x_1, x_2, \cdots, G_{S-1}(x_1,\ldots,x_{S-2})) = \varphi_{lS-2}(x_1, x_2, \cdots, x_{S-2}) = 0,\ l = 1, 2, \ldots, L, \\ \psi_{mS-1}(x_1, x_2, \cdots, G_{S-1}(x_1,\ldots,x_{S-2})) = \psi_{mS-2}(x_1, x_2, \cdots, x_{S-2}) \leq 0,\ m = 1, 2, \ldots, M \end{array}\right\}$$

Multistage optimization

Step S-1. $x_1^* \to F_1\left(x_1^*\right) = \min\limits_{x_1 \in \mathcal{D}_{x1}} F_1\left(x_1\right)$

We may now return to expressions „*G*” determined in the previous steps

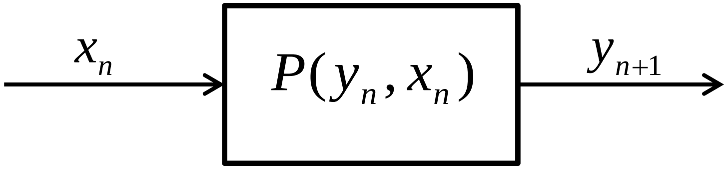
$$x_1^*$$

$$x_2^* = G_2\left(x_1^*\right)$$

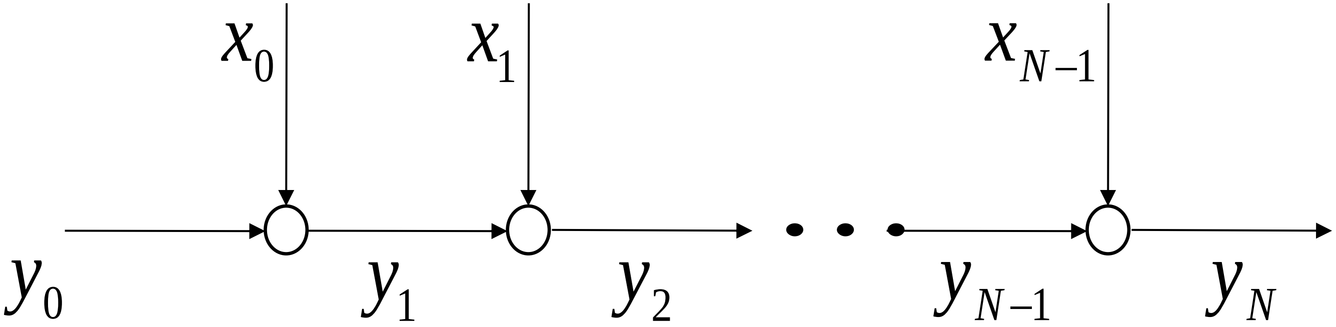
$$\vdots$$

$$x_{S-1}^* = G_{S-1}\left(x_1^*, x_2^* \ldots, x_{S-1}^*\right)$$

$$x_S^* = G_S\left(x_1^*, x^*, \ldots, x_{S-1}^*\right)$$



$$y_{n+1} = P(y_n, x_n),\ y_o$$



$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \overset{\Delta}{=} F(y_0, x_0, x_1, \ldots, x_{N-1})$$

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Dynamic programming

Step 2. $x^*_{N-2} \to \min_{x_{N-2}}\{A_{N-1}(x_{N-2}, y_{N-1}) + V_{N-1}(y_{N-1})\}$

We know, that $y_{N-1} = P(y_{N-2}, x_{N-2})$

$$x^*_{N-2} = G_{N-2}(y_{N-2}) \to \min_{x_{N-2}}\{A_{N-1}(x_{N-2}, P(y_{N-2}, x_{N-2})) + V_{N-1}(P(y_{N-2}, x_{N-2}))\}$$

$$V_{N-2}(y_{N-2}) \overset{\Delta}{=} \min_{x_{N-2}}\{A_{N-1}(x_{N-2}, P(y_{N-2}, x_{N-2})) + V_{N-1}(P(y_{N-2}, x_{N-2}))\} =$$

$$= \{A_{N-1}(x^*_{N-2}, P(y_{N-2}, x^*_{N-2})) + V_{N-1}(P(y_{N-2}, x^*_{N-2}))\} =$$

$$= A_{N-1}(G_{N-2}(y_{N-2}), P(y_{N-2}, G_{N-2}(y_{N-2}))) + V_{N-1}(P(y_{N-2}, G_{N-2}(y_{N-2})))$$

⋮

Dynamic programming

Step N. $$x_0^* \to \min_{x_0}\{A_1(x_0, y_1) + V_1(y_1)\}$$

We know, that $$y_1 = P(y_0, x_0)$$

$$x_0^* = G_0(y_0) \to \min_{x_0}\{A_1(x_0, P(y_0, x_0)) + V_1(P(y_0, x_0))\}$$

y_0 is known and from now on successive decisions may be determined

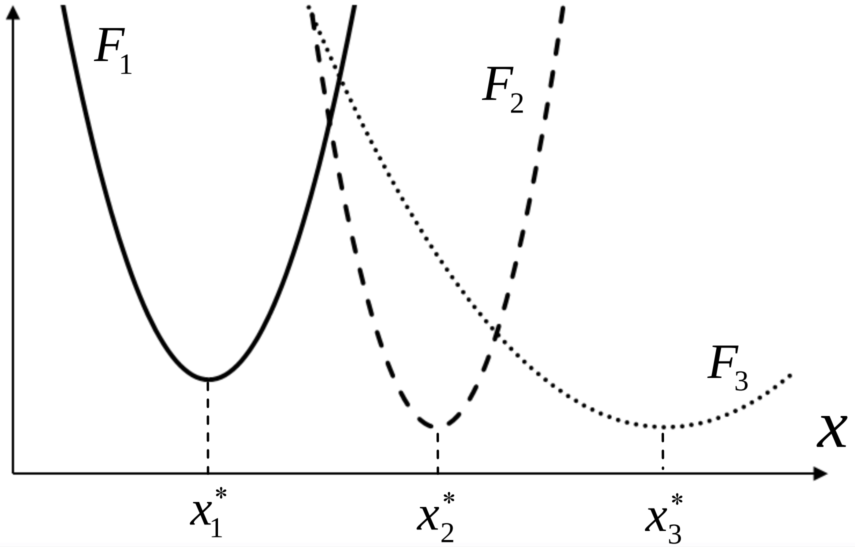
$$x_0^*, x_1^*, \dots, x_{N-1}^*,$$

$$x_0^* = G_0(y_0) \to y_1 = P(y_0, x_0^*)$$
$$x_1^* = G_1(y_1) \to y_2 = P(y_2, x_2^*)$$
$$\vdots$$
$$x_{N-2}^* = G_{N-2}(y_{N-2}) \to y_{N-1} = P(y_{N-2}, x_{N-2}^*)$$
$$x_{N-1}^* = G_{N-1}(y_{N-1}) \to y_N = P(y_{N-1}, x_{N-1}^*)$$

Multicriteria optimization

x – decision variables vector

$F_1(x), F_2(x), \ldots, F_K(x)$ – performance indices



Multicriteria optimization

Synthetic performance index

H(.) – monotonic for all variables

$$F(x) = H\big(F_1(x), F_2(x), \ldots, F_K(x)\big)$$

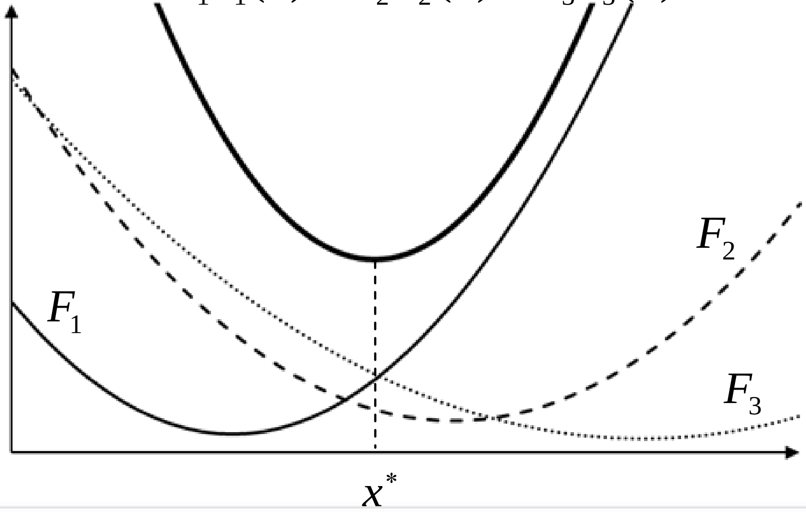
e. g.: $$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x)$$

where: $$\sum_{k=1}^{K} \alpha_k = 1, \;\; \alpha_k > 0, \;\; k = 1,2,\ldots,K$$

$$F(x) = \prod_{k=1}^{K} F_k(x)$$

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

$$F = \alpha_1 F_1(x) + \alpha_2 F_2(x) + \alpha_3 F_3(x)$$



Multicriteria optimization

A selected performance index is optimized,

Upper limits for values of another performance indices are specified.

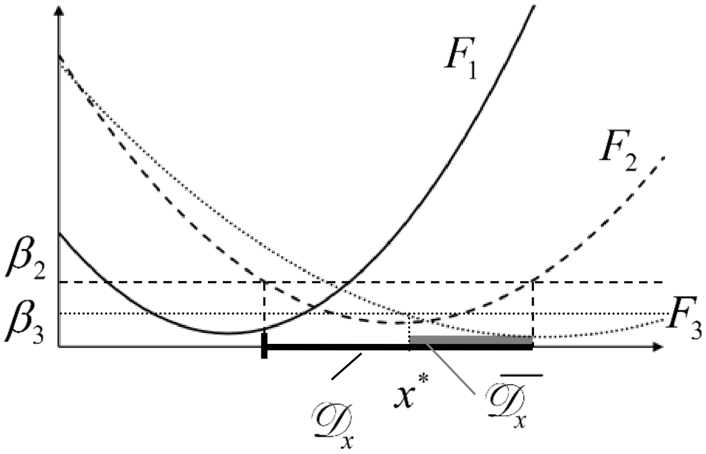
Let $F_1(x)$ be a selected performance index

$$F_k(x) \le \beta_k, \quad k = 2,3,\ldots,K$$

Requirements for performance indices are met

$$\overline{\mathscr{D}_x} = \mathscr{D}_x \cap \left\{ x \in \mathscr{R}^S : F_k(x) \le \beta_k,\ k = 2,\ldots,K \right\}$$

$$x^* \to F_1(x^*) = \min_{x \in \overline{\mathscr{D}_x}} F_1(x)$$



Multicriteria optimization

Ranked/prioritized performance indices

$$F_1(x) \succ F_2(x) \succ \ldots \succ F_K(x) \quad x \in \mathcal{D}_x$$

Step 1. $\mathcal{D}_{x1} = \mathcal{D}_x$

$$x_1^* \to F_1(x_1^*) = \min_{x \in \mathcal{D}_{x1}} F_1(x)$$

Step 2. $\mathcal{D}_{x2} = \mathcal{D}_{x1} \cap \left\{ x \in \mathcal{R}^S : F_1(x) \le F_1\left(x_1^*\right) + \gamma_1 \right\}$

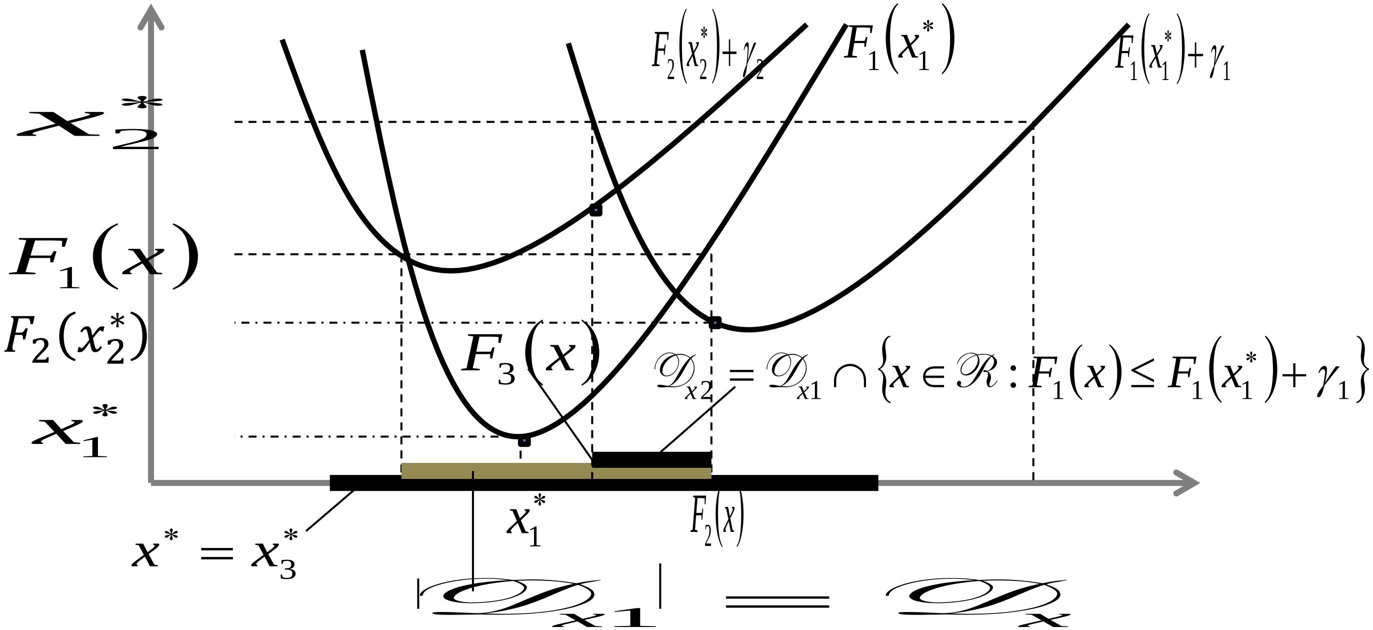
$$x_2^* \to F_2(x_2^*) = \min_{x \in \mathcal{D}_{x2}} F_2(x)$$

$\vdots$

Multicriteria optimization

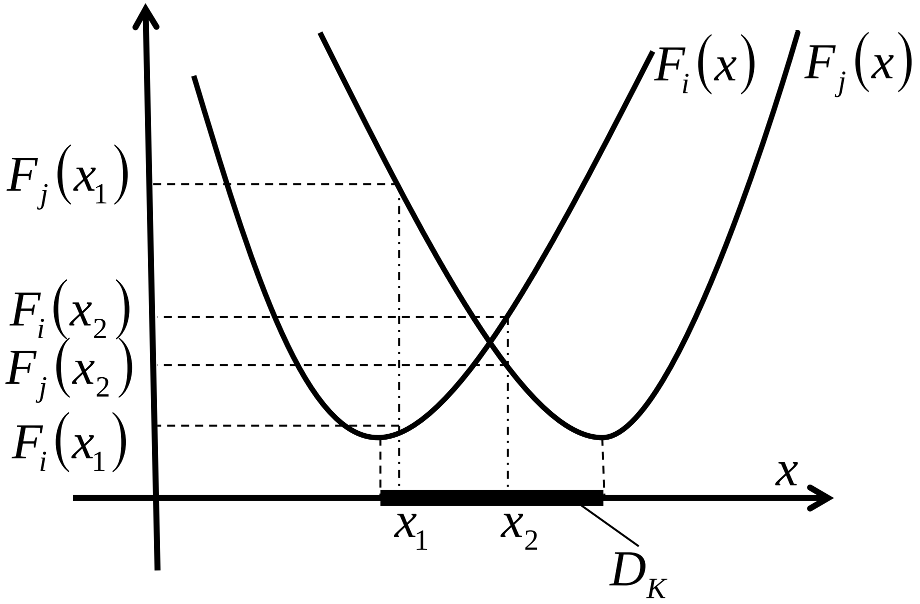
Step K. $\mathscr{D}_{xK} = \mathscr{D}_{xK-1} \cap \left\{x \in \mathscr{R}^S : F_{K-1}(x) \le F_1\left(x^*_{K-1}\right) + \gamma_{K-1}\right\}$

$$x^*_K = x^*_K \to F_K(x^*_K) = \min_{x \in \mathscr{D}_{xK}} F_K(x)$$



Non-dominated solutions

$$x_1, x_2 \in D_K \Leftrightarrow \forall j \in \{1, 2, \ldots, K\} \, \exists \, i \in \{1, 2, \ldots, K\}$$
$$F_j(x_1) > F_j(x_2) \Rightarrow F_i(x_1) < F_i(x_2)$$



Exam

- Term 0: 22.06.2026. (Monday)
  room 23, building C-3, time: 9^{15}-11^{00}
- Term 1: 29.07. 2026. (Monday)
  room 22, building C-3, time: 9^{15}-11^{00}
- Term 2: 6.07. 2026. (Monday)
  room 22, building C-3, time: 9^{15}-11^{00}

Term „zero”- necessary conditions

- Positive grades from practice (classes) and laboratory i.e. ≥3.0 not later then „zero” term
- Final grade proposition mean value integer number i.e.:
- $Final\ grade = \frac{[practice\ (classes) + laboratory]}{2} \geq 3.5$
- Must be present during „zero” term (otherwise reject bonus)

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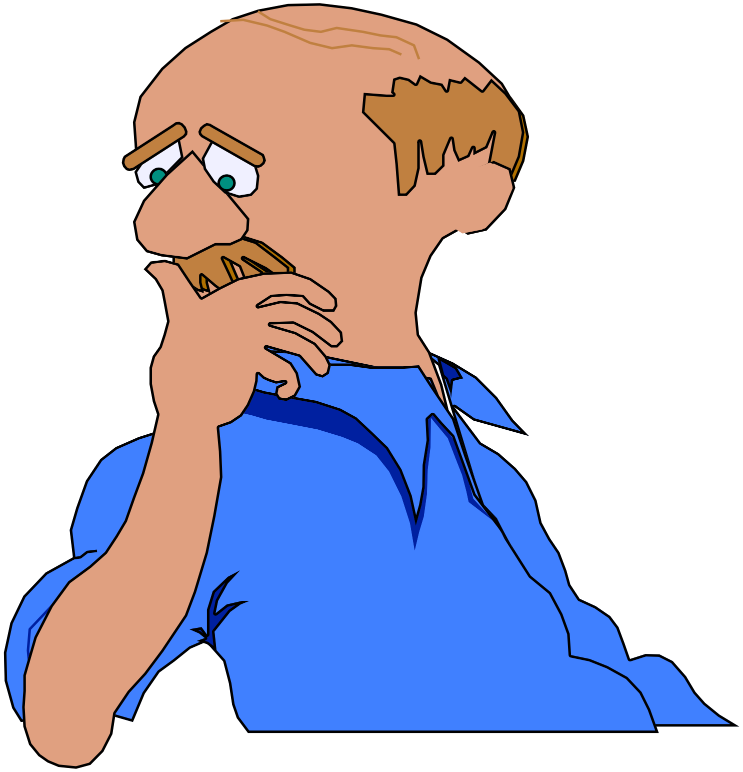
- About marks from this semester I will be informed by my assistants.
- About marks from previous years you must inform me by mail sending positive mark form JSOS (USOS) system with name of teacher, name of student and index number.

Thank you for attention



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