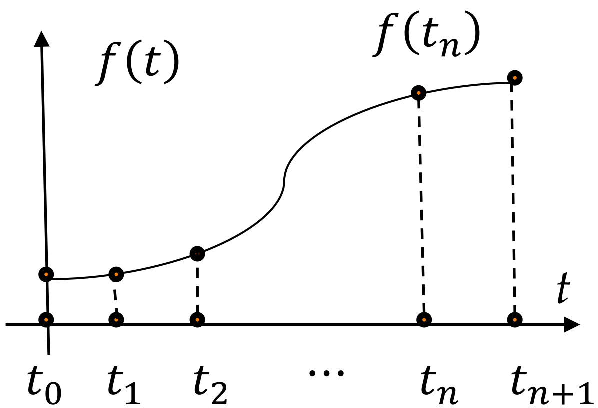
Modele systemów dynamicznych

Wykład 7. Dyskretyzacja sygnałów ciągłych. Opis obiektu ciągłego sterowanego sygnałem dyskretnym

Dyskretyzacja sygnału ciągłego

$f(t), f(t) = 0 \; dla \; t < 0$



$$t_n = t_0 + n\Delta t$$
$$t_0 = 0, n = 0, 1, 2, \cdots$$

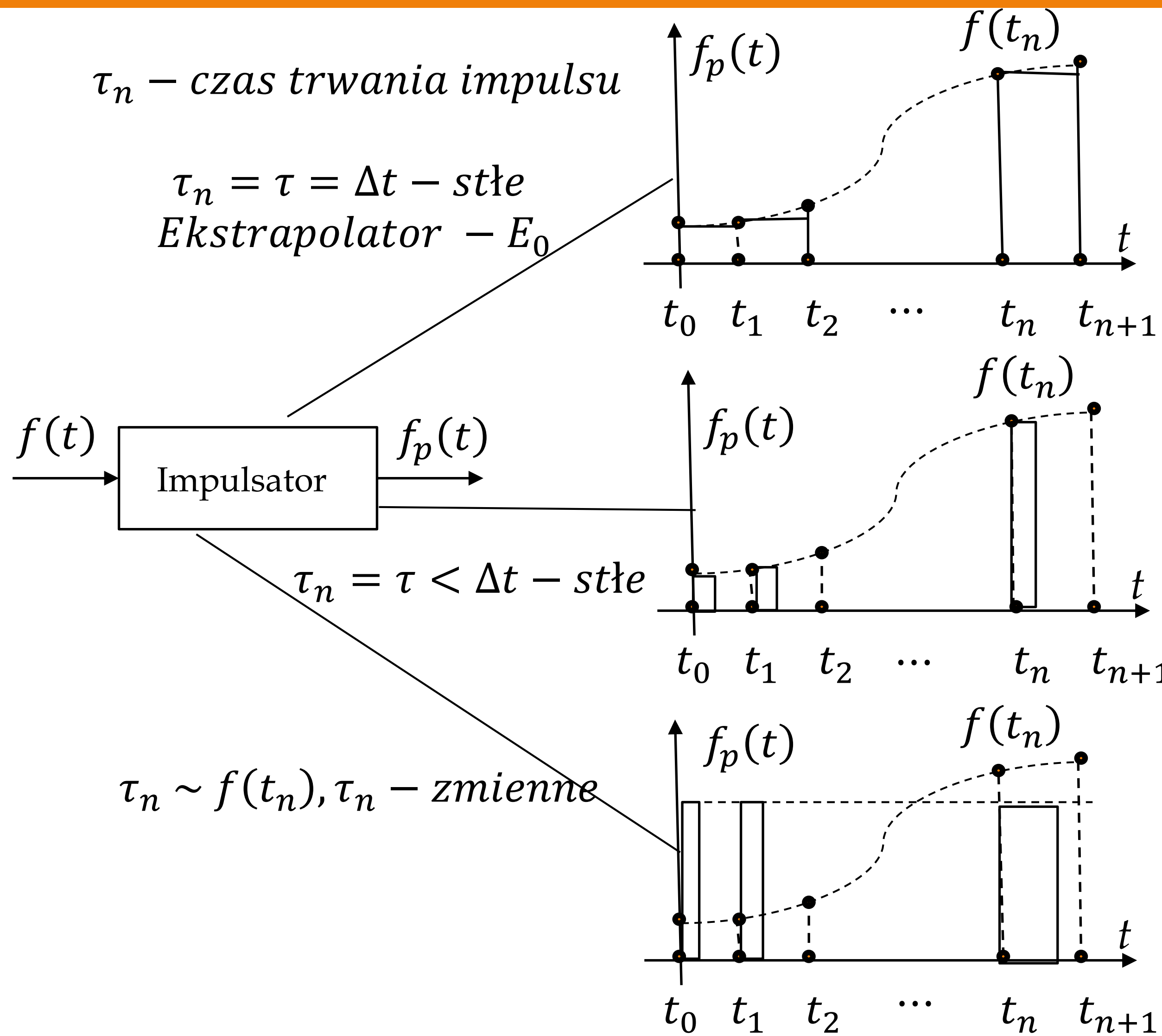
Δt – okres dyskretyzacji

$\tau_n - czas \; trwania \; impulsu$

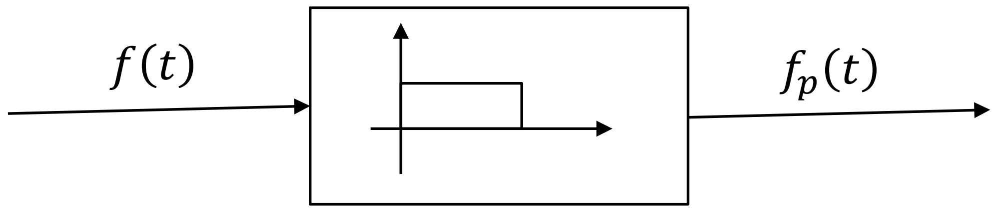
$\tau_n = \tau = \Delta t - stłe$
$Ekstrapolator \; - E_0$

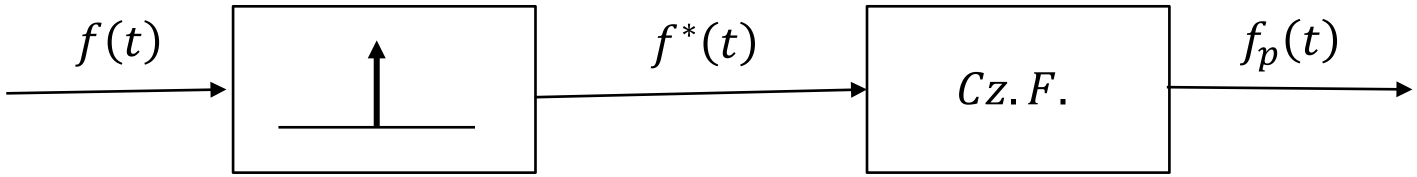
$\tau_n = \tau < \Delta t - stłe$

$\tau_n \sim f(t_n), \tau_n - zmienne$



Iplsator

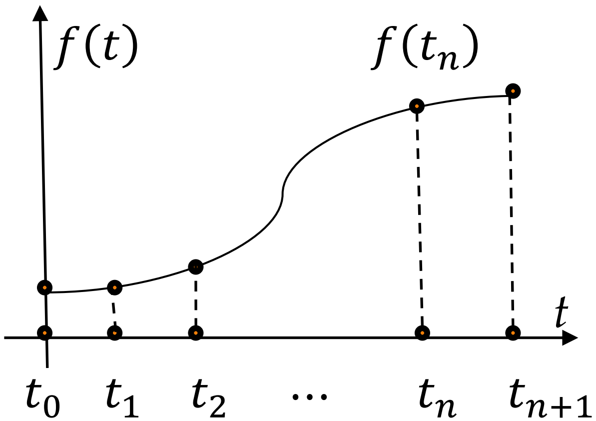


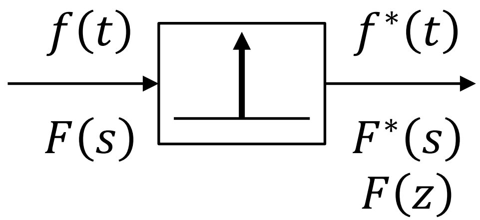


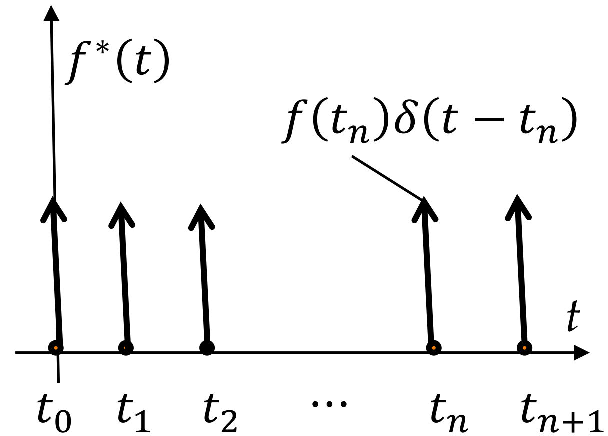
Impulsator idealny

Człon Formułujący

Impulsator idealny







$$f^*(t) \sim \textit{ciąg impulsów } f(t_0)\delta(t-t_0), f(t_1)\delta(t-t_1), \cdots, f(t_n)\delta(t-t_n)$$

Oznaczmy przez: $S(t) = \sum_{n=0}^{\infty} \delta(t - t_n)$

Wówczas:

$$f^*(t) = f(t)S(t) = f(t)\sum_{n=0}^{\infty} \delta(t - t_n) = \sum_{n=0}^{\infty} f(t)\delta(t - t_n)$$

Relacja pomiędzy *F(s)* i *F(z)* *impulsator idealny*

$f(t)$ *zamieniono na ciąg impulsów* $f(t_0)\delta(t-t_0), f(t_1)\delta(t-t_1), \cdots, f(t_n)\delta(t-t_n)$

$f(t)$ *po impulsatorze idealnym* $f^*(t) = f(t)S(t) = f(t)\sum_{n=0}^{\infty}\delta(t-t_n) = \sum_{n=0}^{\infty} f(t)\delta(t-t_n)$

$$F^*(s) = \mathscr{L}\left[f^*(t)\right] = \int_0^\infty \sum_{n=0}^{\infty} f(t)\delta(t-t_n)e^{-st}dt = \sum_{n=0}^{\infty}\int_0^\infty f(t)\delta(t-t_n)e^{-st}dt = \sum_{n=0}^{\infty} f(t_n)^{-st_n}$$

$ponieważ: t_n = t_0 + n\Delta t, t_0 = 0 \; zatem: t_n = n\Delta t \; stąd \; :$

$$F^*(s) = \sum_{n=0}^{\infty} f(t_n)^{-st_n} = \sum_{n=0}^{\infty} f(n\Delta t)e^{-sn\Delta t}$$

$oznaczmy \; przez: f(n\Delta t) \overset{df}{=} f_n, \; e^{s\Delta t} \overset{df}{=} z \qquad stąd$

$$F(s) \xrightarrow{po\ dyskretyzacji} F^*(s) = \sum_{n=0}^{\infty} f(n\Delta t)e^{-sn\Delta t} \longrightarrow \sum_{n=0}^{\infty} f_n z^{-n} = F(z)$$

Relacja pomiędzy *F(s)* i *F(z)*
impulsator idealny

$$F(s) \to f(t) = \mathscr{L}^{-1}\left[F(s)\right] \to f(n\Delta t) = f_n \to F(z) = \mathscr{Z}\left[f_n\right]$$

$$f(t) = \delta(t) = \begin{cases} \infty \; dla \; t = 0 \\ 0 \; dla \; t \neq 0 \end{cases} \to f_n = \delta(n\Delta t) = \delta_n = \begin{cases} 1 \; dla \; n = 0 \\ 0 \; dla \; n \neq 0 \end{cases}$$

$$F(z) = \mathscr{Z}\left[\delta_n\right] = \sum_{n=0}^{\infty} \delta_n z^{-n} = 1$$

$$f(t) = 1(t) = \begin{cases} 1 \; dla \; t \geq 0 \\ 0 \; dla \; t < 0 \end{cases} \to f_n = 1(n\Delta t) = 1_n = \begin{cases} 1 \; dla \; n \geq 0 \\ 0 \; dla \; n < 0 \end{cases}$$

$$F(z) = \mathscr{Z}\left[1_n\right] = \sum_{n=0}^{\infty} 1_n z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Relacja pomiędzy *F(s)* i *F(z)* *impulsator idealny*

$$f(t)=1(t)e^{at} \rightarrow f_n = f(n\Delta t)=1_n e^{an\Delta t} = 1_n\left(e^{a\Delta t}\right)^n$$

$$F(z)=\mathscr{Z}\left[1_n\left(e^{a\Delta t}\right)^n\right]=\sum_{n=0}^{\infty}\left(e^{a\Delta t}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{e^{a\Delta t}}{z}\right)^n = \frac{1}{1-\frac{e^{a\Delta t}}{z}} = \frac{z}{z-e^{a\Delta t}}$$

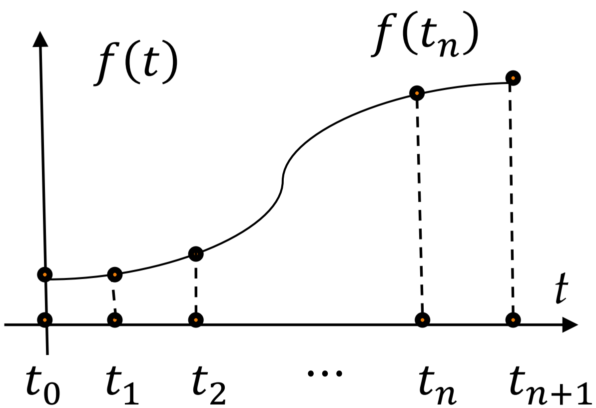
$$f(t)=1(t)t \qquad \rightarrow f_n = 1_n n\Delta t \qquad \rightarrow F(z)=\frac{\Delta t z}{(z-1)^2}$$

$$f(t)=1(t)\sin\beta t \rightarrow f_n = 1_n \sin\beta n\Delta t \rightarrow F(z)=\frac{z\sin\beta\Delta t}{z^2-2\cos\beta\Delta t+1}$$

$$f(t)=1(t)\cos\beta t \rightarrow f_n = 1_n \cos\beta n\Delta t \rightarrow F(z)=\frac{z^2-2z\cos\beta\Delta t}{z^2-2\cos\beta\Delta t+1}$$

Dyskretyzacja sygnału ciągłego

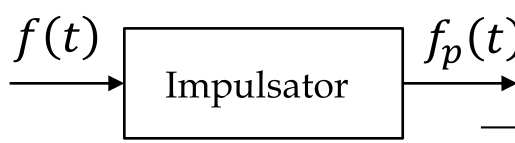
$f(t), f(t) = 0 \; dla \; t < 0$



$$t_n = t_0 + n\Delta t$$
$$t_0 = 0$$

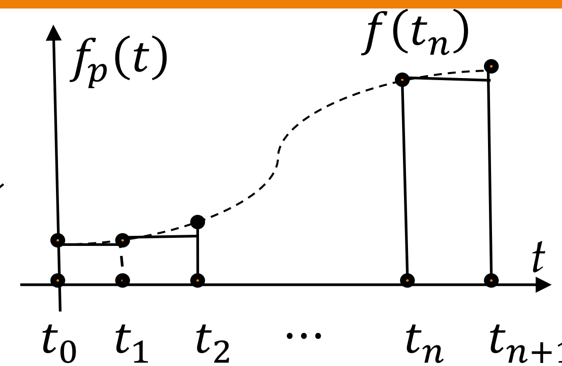
$\tau_n - czas \; trwania \; impulsu$

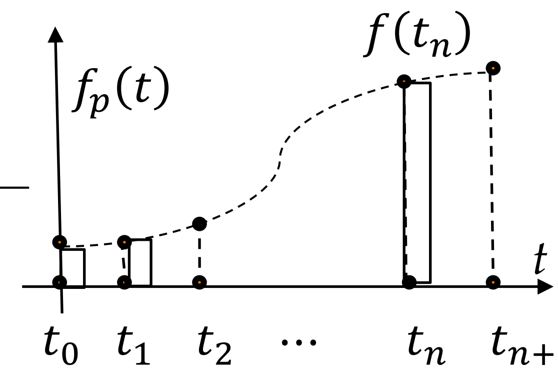
$\tau_n = \tau = \Delta t - stałe$
$Ekstrapolator \; - E_0$

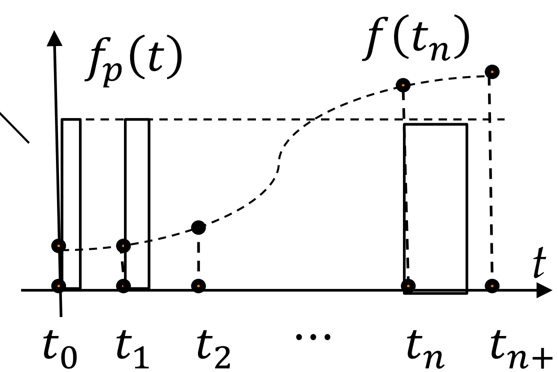


$\tau_n = \tau < \Delta t - stałe$

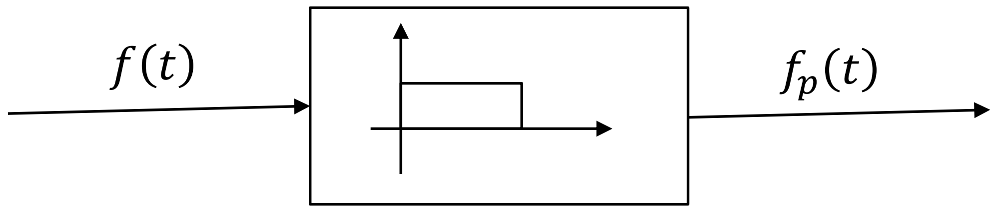
$\tau_n \sim f(t_n), \tau_n - zmienne$

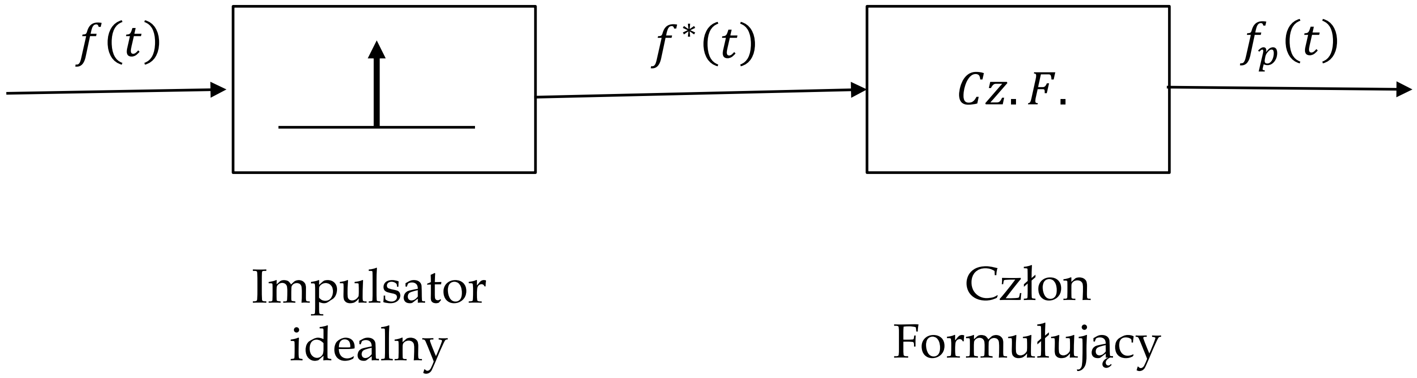






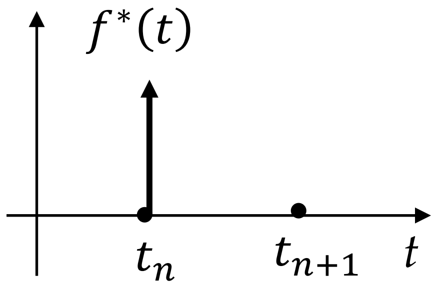
Iplsator



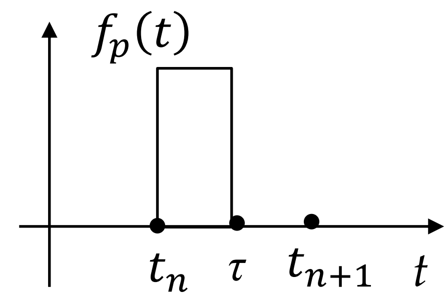


Człon formułujący
szerokość impulsu – τ<Δt - stałe

$$f^*(t) = f(t_n)\delta(t-t_n) \longrightarrow \boxed{Cz.F.} \longrightarrow f_p(t) = f(t_n)\big(1(t-t_n) - 1(t-t_n-\tau)\big)$$



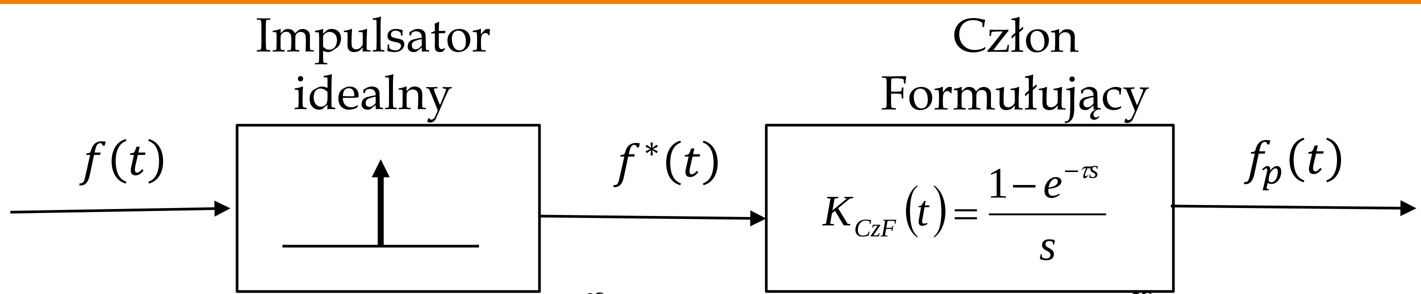
$$K_{CzF}(t) = \frac{1-e^{-\tau s}}{s} \overset{\tau=\Delta t}{=} \frac{1-e^{-\Delta t s}}{s} = E_0(s)$$



$$K_{CzF}(s) = \frac{\mathcal{L}\left[f_p(t)\right]}{\mathcal{L}\left[f^*(t)\right]} = \frac{\mathcal{L}\left[f(t_n)\big(1(t-t_n) - 1(t-t_n-\tau)\big)\right]}{\mathcal{L}\left[f(t_n)\delta(t-t_n)\right]} =$$

$$= \frac{f(t_n)\left(\frac{1}{s}e^{-t_n s} - \frac{1}{s}e^{(-t_n-\tau)s}\right)}{f(t_n)e^{-t_n s}} = \frac{1}{s} - \frac{1}{s}e^{-\tau s} = \frac{1-e^{-\tau s}}{s}$$

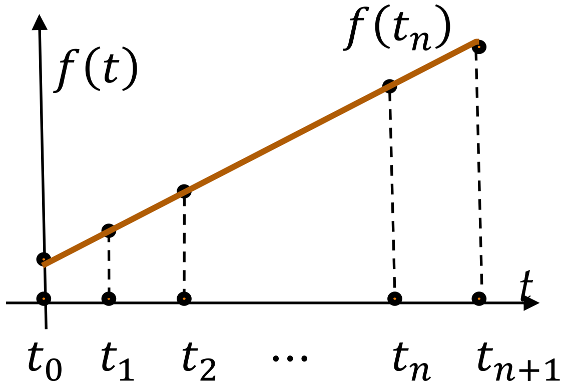
Impulsator – $\tau<\Delta t$ - stałe

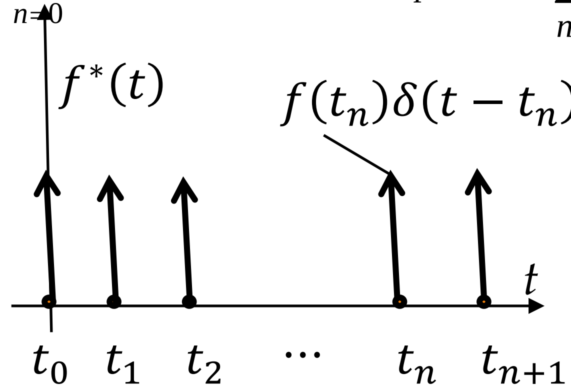


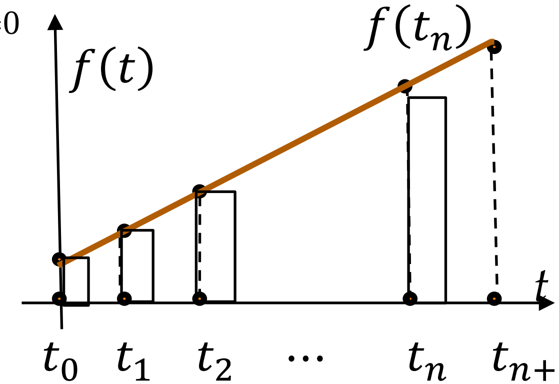
$f(t)$

$$f^*(t)=\sum_{n=0}^{\infty} f(t_n)\delta(t-t_n)$$

$$f_p(t)=\sum_{n=0}^{\infty} f(t_n)\big(1(t-t_n)-1(t-t_n-\tau)\big)$$

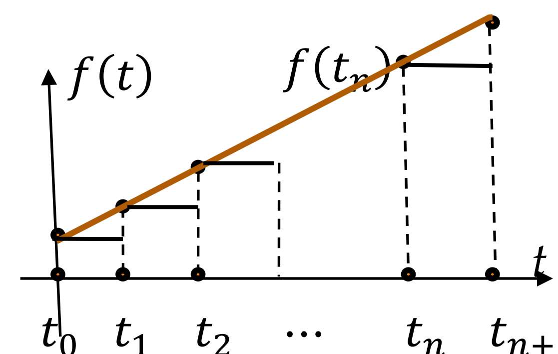






Szczególny przypadek $\tau=\Delta t$ - Eksplorator rzędu zerowego

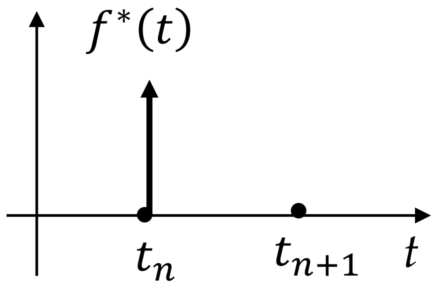
$$K_{CzF}(t)=\frac{1-e^{-\tau s}}{s}\overset{\tau=\Delta t}{=}\frac{1-e^{-\Delta t s}}{s}=E_0(s)$$



Człon formułujący
szerokość impulsu – $\tau_n<\Delta t$ - zmienne

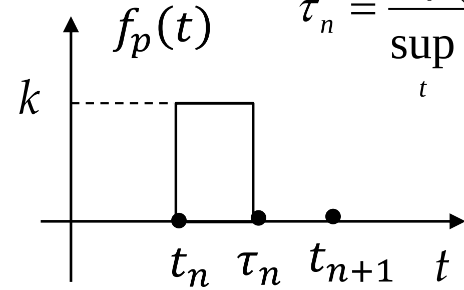
$$f^*(t)=f(t_n)\delta(t-t_n)$$

$$f_p(t)=k\left(1(t-t_n)-1(t-t_n-\tau_n)\right)$$



$$K_{CzF}(s)=\frac{k}{f(t_n)}\frac{1-e^{-\tau_n s}}{s}$$

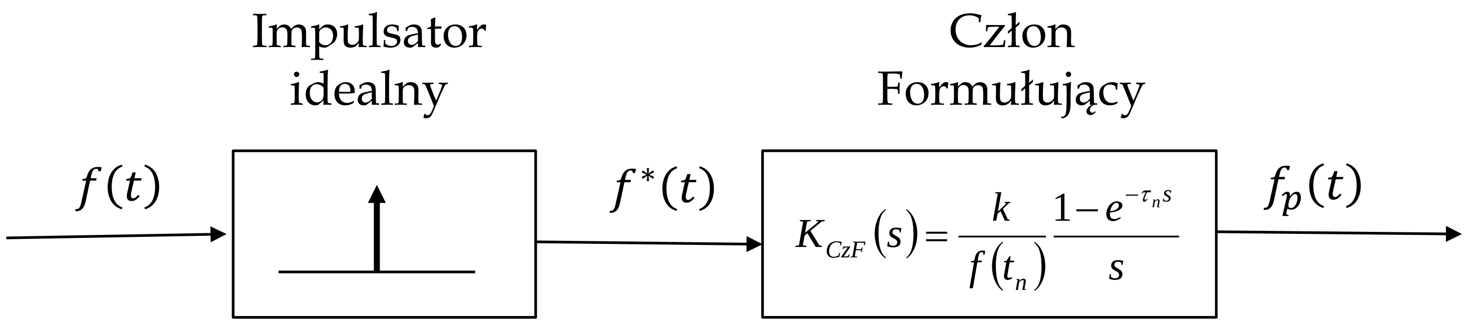
$$\tau_n=\frac{f(t_n)}{\sup_t f(t)}$$



$$K_{CzF}(s)=\frac{\mathscr{L}\left[f_p(t)\right]}{\mathscr{L}\left[f^*(t)\right]}=\frac{\mathscr{L}\left[k\left(1(t-t_n)-1(t-t_n-\tau_n)\right)\right]}{\mathscr{L}\left[f(t_n)\delta(t-t_n)\right]}=$$

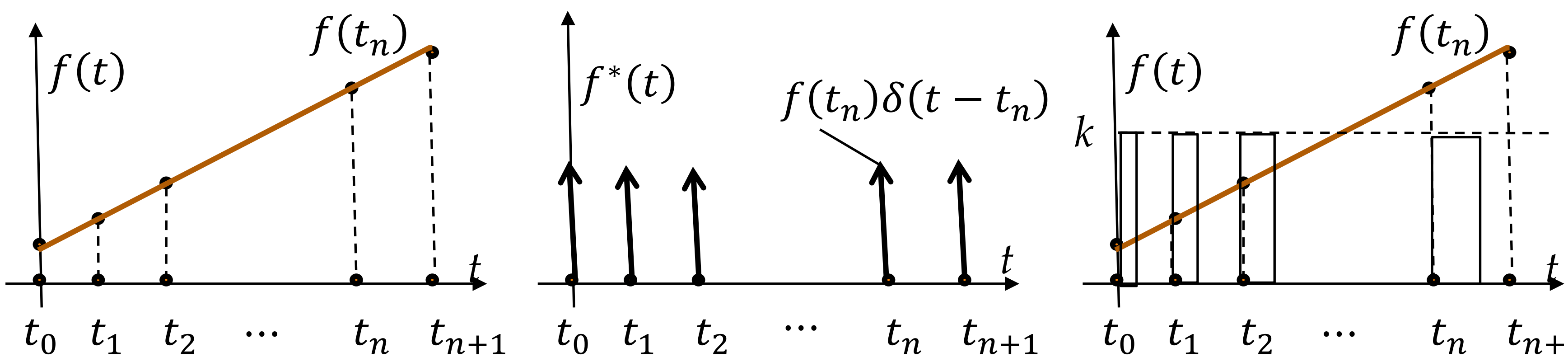
$$=\frac{k\left(\frac{1}{s}e^{-t_n s}-\frac{1}{s}e^{(-t_n-\tau_n)s}\right)}{f(t_n)e^{-t_n s}}=\frac{k}{f(t_n)}\left(\frac{1}{s}-\frac{1}{s}e^{-\tau_n s}\right)=\frac{k}{f(t_n)}\frac{1-e^{-\tau_n s}}{s}$$

Impulsator – $\tau_n < \Delta t$ - zmienne



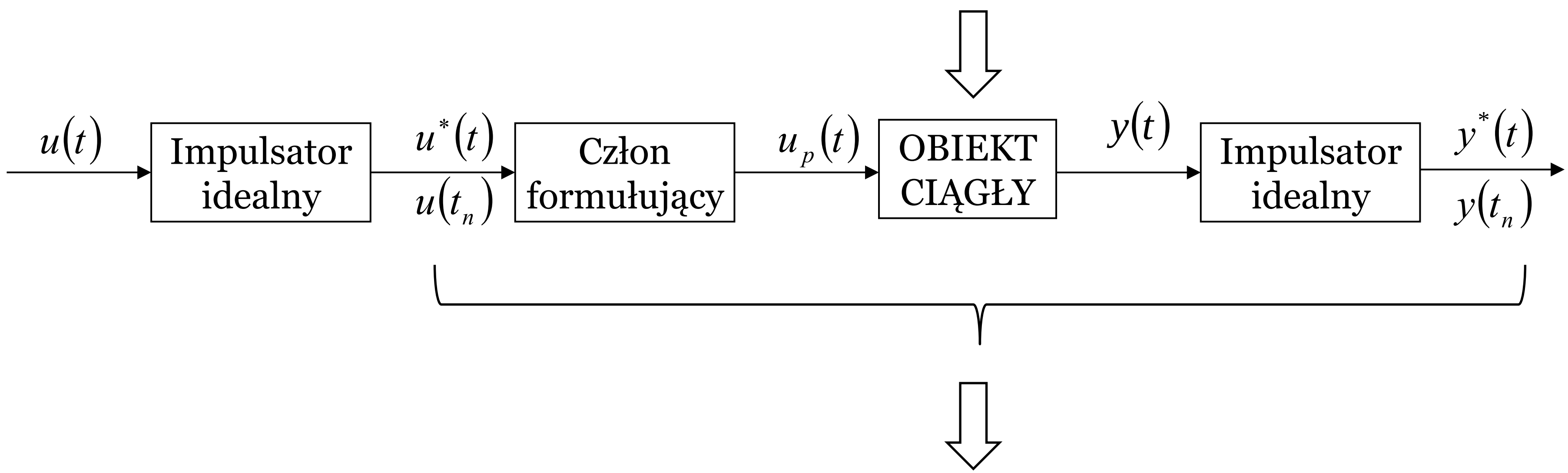
$$f(t)$$

$$f^*(t) = \sum_{n=0}^{\infty} f(t_n)\delta(t - t_n) \qquad f_p(t) = \sum_{n=0}^{\infty} f(t_n)\big(1(t - t_n) - 1(t - t_n - \tau_n)\big)$$



Relacje pomiędzy opisami
Równania stanu

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

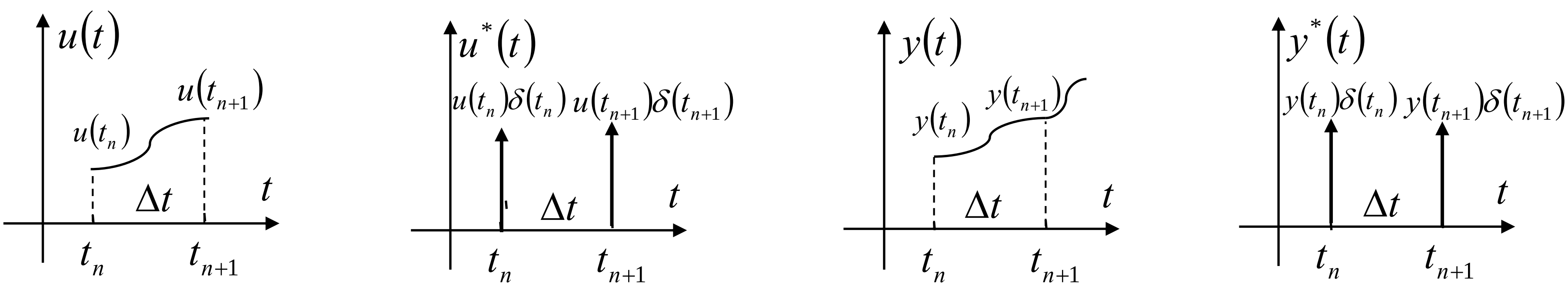


$$x_{n+1} = \overline{A}x_n + \overline{B}u_n$$
$$y_n = \overline{C}x_n + \overline{D}u_n$$

Relacje pomiędzy opisami
Równania stanu

Opis przy pomocy równań stanu – impulsator idealny bez członu formułującego

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x_0 = x(t_0), \quad y(t) = Cx(t) + Du(t)$$



Relacje pomiędzy opisami
Równania stanu

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x_0 = x(t_0), \quad y(t) = Cx(t) + Du(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)}Bu^*(\tau)d\tau, \quad u^*(\tau) = u(t_n)\delta(t-t_n)$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)}Bu(t_n)\delta(\tau-t_n)d\tau$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + e^{A(t_{n+1}-t_n)}Bu(t_n) = e^{A\Delta t}x(t_n) + e^{A\Delta t}Bu(t_n)$$

niech: $x(t_n) = x_n, \quad u_n = u(t_n), \quad y_n = y(t_n)$

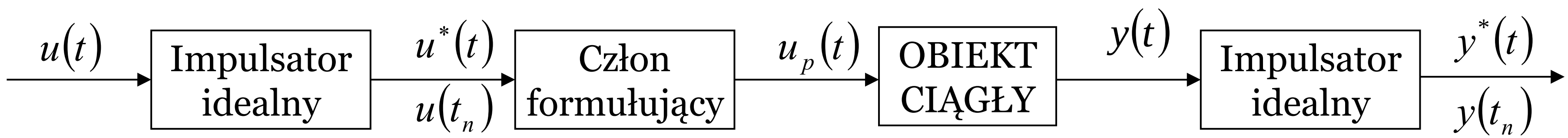
$$\begin{aligned} x_{n+1} &= e^{A\Delta t}x_n + e^{A\Delta t}Bu_n \\ y_n &= Cx_n + Du_n \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_{n+1} &= \overline{A}x_n + \overline{B}u_n \\ y_n &= \overline{C}x_n + \overline{D}u_n \end{aligned}$$

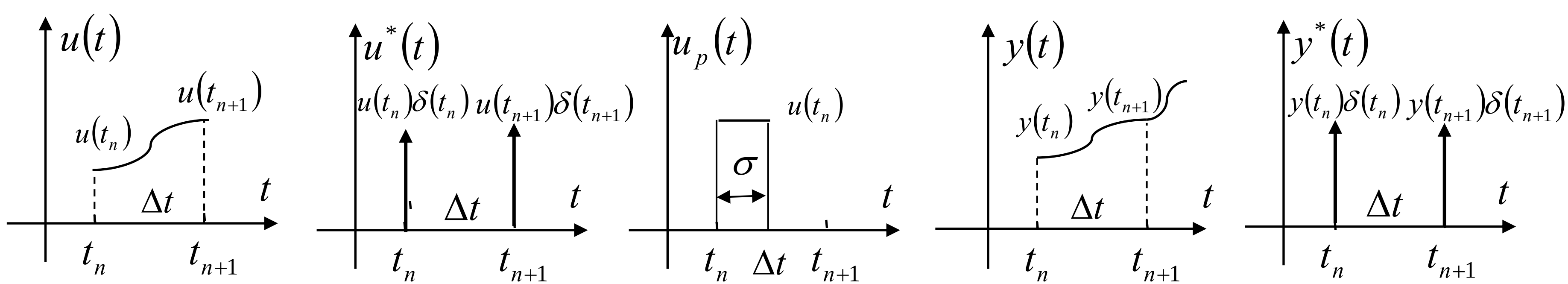
gdzie: $\overline{A} = e^{A\Delta t}, \quad \overline{B} = e^{A\Delta t}B, \quad \overline{C} = C, \quad \overline{D} = D$

Relacje pomiędzy opisami
Równania stanu

Opis przy pomocy równań stanu – impulsator z członem formułującym

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x_0 = x(t_0), \quad y(t) = Cx(t) + Du(t)$$





Relacje pomiędzy opisami
Równania stanu

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_n-\tau)}Bu^*(\tau)d\tau, \quad u^*(\tau) = \begin{cases} u(t_n) & dla\ t_n \le t \le t_n + \sigma \\ 0 & dla\ t_n + \sigma \le t \le t_{n+1} \end{cases}$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_n+\sigma} e^{A(t_{n+1}-\tau)}Bu(t_n)d\tau = e^{A\Delta t}x(t_n) + \int_{t_n}^{t_n+\sigma} e^{A(t_n+\Delta t-\tau)}Bu(t_n)d\tau$$

$$niech: \quad \bar{\tau} = \Delta t - \sigma \qquad x(t_{n+1}) = e^{A\Delta t}x(t_n) + \int_{\Delta t}^{\Delta t-\sigma} e^{A\bar{\tau}}Bd\bar{\tau}\, u(t_n)$$

$$niech: \qquad x(t_n) = x_n, \quad u_n = u(t_n), \quad y_n = y(t_n)$$

$$x_{n+1} = e^{A\Delta t}x_n + u_n \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau}\ u_n$$

$$y_n = Cx_n + Du_n$$

$$\Longrightarrow \qquad x_{n+1} = \bar{A}x_n + \bar{B}u_n, \qquad y_n = \bar{C}x_n + \bar{D}u_n$$

$$gdzie: \qquad \bar{A} = e^{A\Delta t}, \quad \bar{B} = \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau}, \quad \bar{C} = C, \quad \bar{D} = D$$

Szczególny przypadek: $\sigma = \Delta t$ ***Ekstrapolator rzędu zerowego -*** E_0

$$\bar{B} = \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau} = \int_{0}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau} = A^{-1}\left[e^{A\Delta t} - I\right]B$$

Relacje pomiędzy opisami
Człon inercyjny pierwszego rzędu

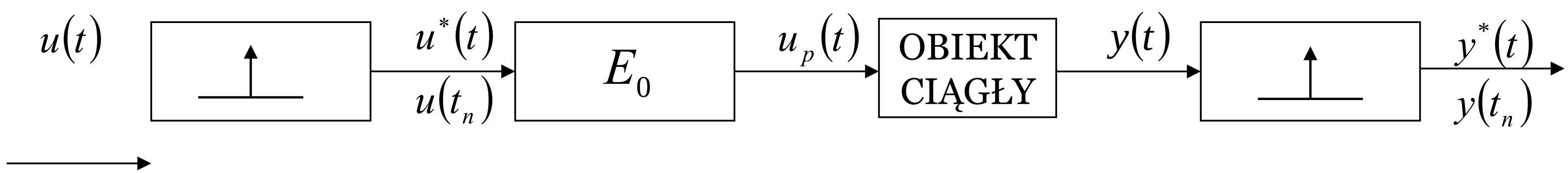
Transmitancja

$$K(s)=\frac{k}{sT+1}$$

Równania stanu

$$\frac{dx(t)}{dt}=Ax(t)+Bu(t)=-\frac{1}{T}x(t)+\frac{k}{T}u(t)$$
$$y(t)=Cx(t)+Du(t)=x(t)+0u(t)$$

$$A=-\frac{1}{T},\quad B=\frac{k}{T}$$
$$C=1,\quad D=0$$



Dla impulsatora z członem formułującym w postaci ekstrapolatora zerowego rzędu

$$\overline{A}=e^{A\Delta t}=e^{-\frac{\Delta t}{T}},\quad \overline{B}=A^{-1}\left[e^{A\Delta t}-I\right]B=-T\left(e^{-\frac{\Delta t}{T}}-1\right)\frac{k}{T}=k\left(1-e^{-\frac{\Delta t}{T}}\right),$$

$$\overline{C}=C=1,\quad \overline{D}=D=0$$

$$\frac{dx(t)}{dt}=Ax(t)+Bu(t)=-\frac{1}{T}x(t)+\frac{k}{T}u(t)$$
$$y(t)=Cx(t)+Du(t)=x(t)+0u(t)$$

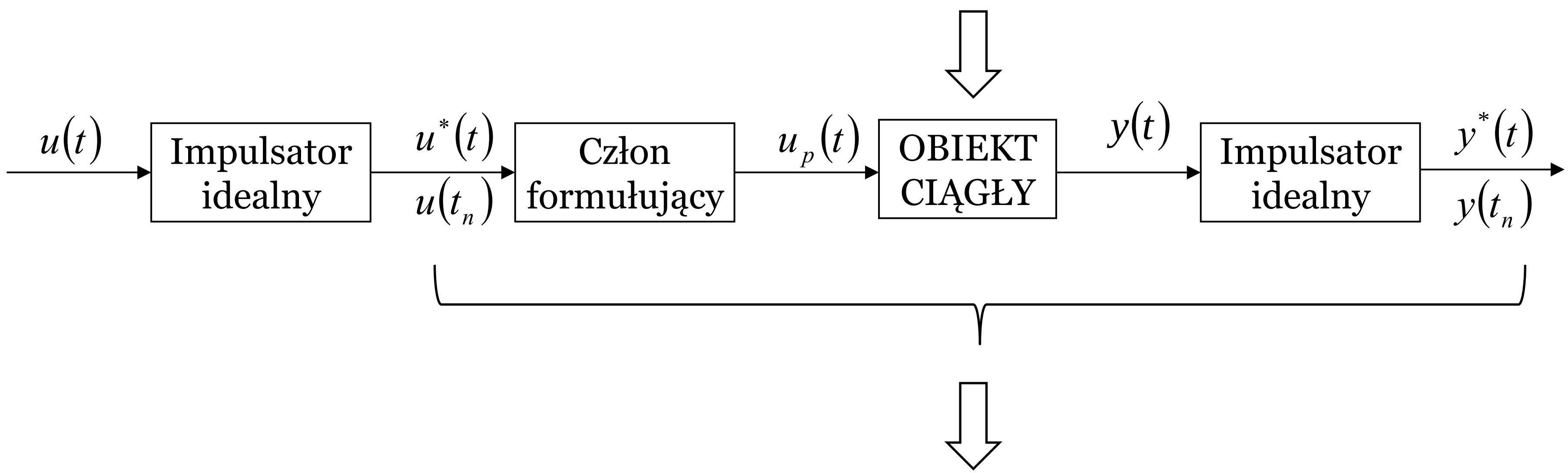
$$\Rightarrow$$

$$x_{n+1}=\overline{A}x_n+\overline{B}u_n=e^{-\frac{\Delta t}{T}}x_n+k\left(1-e^{-\frac{\Delta t}{T}}\right)u_n$$

$$y_n=\overline{C}x_n+\overline{D}u_n=x_n$$

Relacje pomiędzy opisami
Transmitancja obiektu

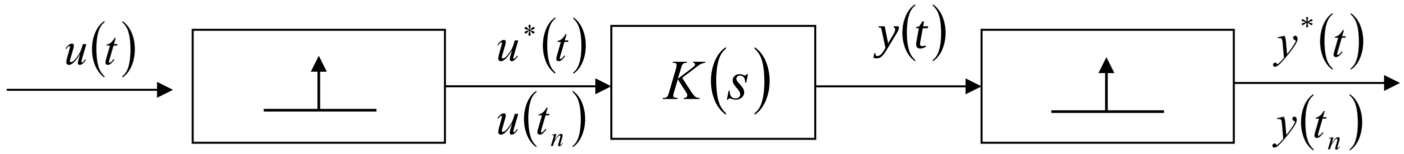
$$K(s)$$



$$\overline{K}(z)$$

Relacje pomiędzy opisami
Transmitancja obiektu

Impulsator idealny (bez członu formułującego)



$$K(s) \quad \Rightarrow \quad k_i(t) = \mathscr{L}^{-1}(K(s)) \quad \Rightarrow \quad k_i(n\Delta t) \quad \Rightarrow \quad \overline{K}(z) = \mathscr{Z}\{k_i(n\Delta t)\}$$

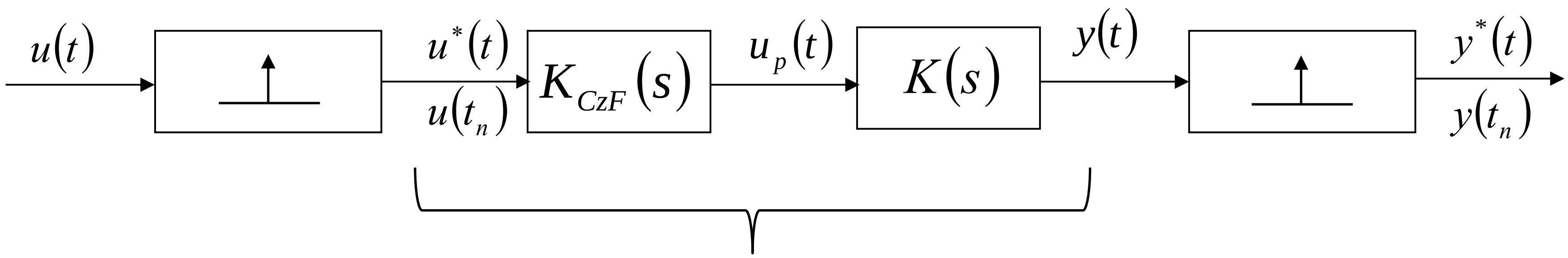
Przykład:

$$K(s) = \frac{k}{1+sT} \quad \Rightarrow \quad k_i(t) = \mathbf{1}(t)\frac{k}{T}e^{-\frac{t}{T}} \quad \Rightarrow \quad k_i(n\Delta t) = \mathbf{1}_{\mathbf{n}}\frac{k}{T}e^{-\frac{n\Delta t}{T}}$$

$$\overline{K}(z) = \mathscr{Z}\{k_i(n\Delta t)\} = \mathscr{Z}\left\{\mathbf{1}_{\mathbf{n}}\frac{k}{T}e^{-\frac{n\Delta t}{T}}\right\} = \frac{k}{T}\frac{z}{z-e^{-\frac{\Delta t}{T}}}$$

Relacje pomiędzy opisami
Transmitancja obiektu

Impulsator z członem formułującym

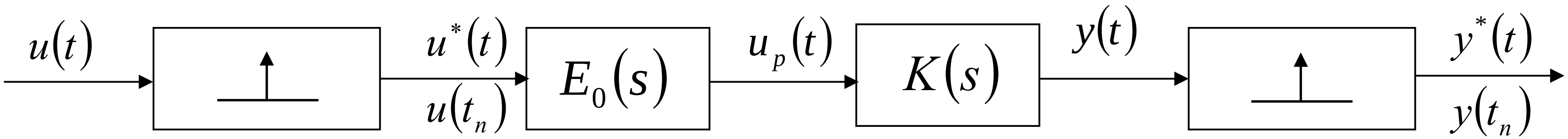


$$K_Z(s) = K_{CzF}(s)K(s)$$

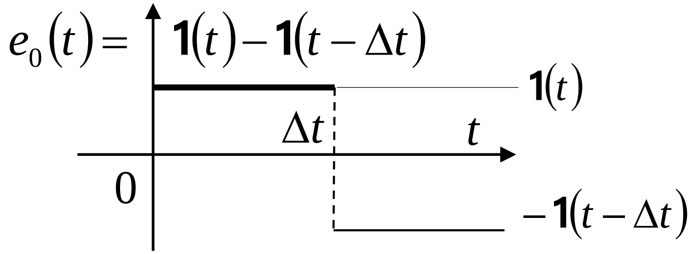
$$K_Z(s) \Rightarrow k_{iZ}(t) = \mathscr{L}^{-1}(K_Z(s)) \Rightarrow k_{iZ}(n\Delta t) \Rightarrow \overline{K}(z) = \mathscr{Z}\{k_{iZ}(n\Delta t)\}$$

Relacje pomiędzy opisami

Transmitancja obiektu - przykład



$$K_{CzF}(s) = E_0(s) = \mathscr{L}(e_0(t)) = \mathscr{L}(\mathbf{1}(t) - \mathbf{1}(t-\Delta t)) = \frac{1-e^{-s\Delta t}}{s} \qquad K(s) = \frac{k}{1+sT}$$



$$K_Z(s) = K_{CzF}(s)K(s) = \frac{1-e^{-s\Delta t}}{s}\,\frac{k}{1+sT}$$

$$k_{iZ}(t) = \mathscr{L}^{-1}\left(\frac{1-e^{-s\Delta t}}{s}\,\frac{k}{1+sT}\right) = \mathbf{1}(t)k\left(1-e^{-\frac{t}{T}}\right) - \mathbf{1}(t-\Delta t)k\left(1-e^{-\frac{t-\Delta t}{T}}\right)$$

$$k_{iZ}(n\Delta t) = \mathbf{1}_{\mathbf{n}}k\left(1-e^{-\frac{n\Delta t}{T}}\right) - \mathbf{1}_{\mathbf{n-1}}k\left(1-e^{-\frac{n\Delta t-\Delta t}{T}}\right) = \mathbf{1}_{\mathbf{n}}k\left(1-e^{-\frac{n\Delta t}{T}}\right) - \mathbf{1}_{\mathbf{n-1}}k\left(1-e^{-\frac{(n-1)\Delta t}{T}}\right)$$

Relacje pomiędzy opisami
Transmitancja obiektu - przykład

$$\overline{K}(z) = \mathscr{Z}\{k_{iZ}(n\Delta t)\} = \mathscr{Z}\left\{\mathbf{1_n} k\left(1-e^{-\frac{n\Delta t}{T}}\right) - \mathbf{1_{n-1}} k\left(1-e^{-\frac{(n-1)\Delta t}{T}}\right)\right\} =$$

$$\mathscr{Z}\left\{\mathbf{1_n} k\left(1-e^{-\frac{n\Delta t}{T}}\right)\right\} - \frac{1}{z}\mathscr{Z}\left\{\mathbf{1_n} k\left(1-e^{-\frac{n\Delta t}{T}}\right)\right\} =$$

$$k\left(\frac{z}{z-1} - \frac{z}{z-e^{-\frac{\Delta t}{T}}}\right) - k\frac{1}{z}\left(\frac{z}{z-1} - \frac{z}{z-e^{-\frac{\Delta t}{T}}}\right) =$$

$$k\left(\frac{z-1}{z-1} - \frac{z-1}{z-e^{-\frac{\Delta t}{T}}}\right) = k\left(1-\frac{z-1}{z-e^{-\frac{\Delta t}{T}}}\right) = k\frac{1-e^{-\frac{\Delta t}{T}}}{z-e^{-\frac{\Delta t}{T}}}$$

Dziękuję za uwagę